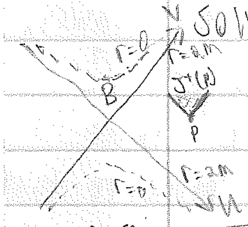
11/13/19

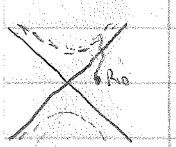
Day 5: Penrose Singularity Theorem, Black Holes, Cosmic Censorship



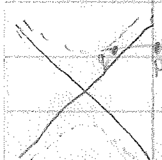
We've discussed the singular nature of the Schwarzschild solution, but not why we might call it a black hole.

Recall that the metric in the u-v coordinate system drawn is such that ∂_u, ∂_v are null and combinations $a\partial_v - b\partial_u$ ($a, b > 0$) are future-pointing timelike, so the set of future-pointing timelike directions at each point span between 45° and 135° from the horizontal. Since the u and v axes correspond to $r=2m$, this observation implies that future-pointing timelike curve beginning in the $v>0$, $r \leq 2m$ region (denoted B) can never leave this region, though they can enter this region if they begin elsewhere. This property, $J^+(B) \subseteq B$, together with the fact that B is "small" in the sense that it is confined to finite r, leads us to call B the black hole region.

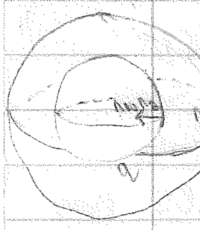
Notice that <u>every</u> future-pointing timelike curve in B reaches $r=0$ in finite proper time. Since Schwarzschild is the unique vacuum spherically symmetric solution, it must be the spacetime exterior to a nonrotating star. It is easy to think that, though Schwarschild is relevant for $r > R_0$ (R_0 the radius of the star), a collapsing star will always have some core that "hides" $r=0$, so the $r=0$ singularity is not physically relevant — this is misguided. Since each point in the u,v diagram is a sphere, the evolution of the surface of the star is given by a future-directed timelike curve, so if we ever have $R_0 \leq 2m$, the surface necessarily reaches $r=0$ in a finite proper time (of an observer on the surface). That is, once $R_0 \leq 2m$, there is no outward force the star's interior could possibly exert to stop it from collapsing fully, eventually "exposing" the $r=0$ singularity.



The Penrose singularity theorem seeks to extract the essential geometric feature of such a system that guarantees the inevitability of singular behavior. Consider a three-dimensional spacelike slice Σ of Schwarzschild as drawn,

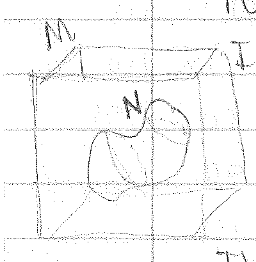


so Σ looks like $\mathbb{R}^3 \setminus \{0\}$. The points p and q on Σ represent spheres centered at the removed origin; observe that the two null directions normal to the sphere at p have the property that one points towards increasing r while the other towards decreasing r, but at q both point towards decreasing r. This is the feature of the sphere at q we'd like to extract.



In general, in the absence of spherical symmetry we do not have a canonical function r indicating size within a family of surfaces, so we need another way of characterizing this shrinking in normal null directions to a compact spacelike 2-dimensional submanifold $N \subseteq M$ playing the role of the Schwarzschild spheres. One such characterization is found in all of the null geodesics normal to N being focusing as in the previous talk; recall that this was ensured for a geodesic $\gamma(s)$ normal to N at $s=0$ by $\langle \gamma'(0), H \rangle > 0$, where H is the mean curvature vector to N. That $\langle n, H \rangle > 0$ for every normal null direction to N is further equivalent to H being past-directed and timelike, and N is called <u>future converging</u> if this is the case (or a <u>trapped surface</u>). A lemma important to interpreting Penrose's theorem is that, if (M, g) is future null-complete and satisfies the weak energy condition, then a compact achronal spacelike $(n-2)$ submanifold N that is future converging has compact $E^+(N) := J^+(N) \setminus I^+(N)$.

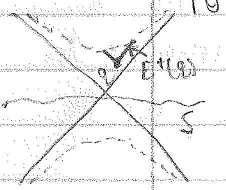
Penrose's result is then:



If (1) the weak energy condition holds
(2) M has a Cauchy hypersurface (is globally hyperbolic)
(3) N is a compact achronal spacelike codimension 2 future converging submanifold
(4) M is future null complete

Then $E^+(N)$ is a Cauchy hypersurface for M.

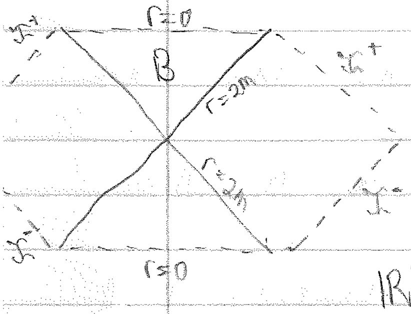
In particular, if (1), (2), and (3) hold and the Cauchy surface of M is noncompact, then M is future null incomplete. Similarly, if (1), (2), and (3) hold and there is an inextendible timelike curve not meeting $E^+(N)$, then M is future null incomplete.



To get a feel for these conditions, consider N as the sphere at q in Schwarzschild as before; $E^+(N) = E^+(q)$ and a Cauchy hypersurface S are as drawn. The properties that S is noncompact and there are many inextendible timelike curves avoiding $E^+(q)$ will not change under perturbations away from spherical symmetry, so the singular behavior is not intrinsically dependent on this symmetry!

In this more general scenario of singular behavior without spherical symmetry, when should we say a black hole is formed? As in the Schwarzschild case, we'd certainly require that the black hole region B satisfy $J^+(B) \subseteq B$, but this is insufficient ($J^+(M) \subseteq M$, for example) — we need to characterize B being "small" and "isolated" in some sense. Toying with Schwarzschild will clearly indicate that the usual topological notion of smallness, compactness, does not fit the bill.

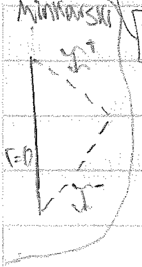
The standard characterization of this isolation that does not depend on the radial function in the Schwarzschild case is to recognize that if one rescales the U, V coordinates



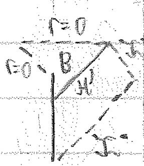
to have finite range, one obtains a Penrose diagram for the spacetime as drawn. The dashed lines labelled $\mathcal{I}^{\pm}$ represent the endpoints of complete inextendible null geodesics in Schwarzschild. Thinking of this diagram as sitting in a surrounding Minkowski $\mathbb{R}^2$, notice that (in the causal structure of this surrounding $\mathbb{R}^2$), we have that $J^-(\mathcal{I}^+)$, the causal past of $\mathcal{I}^+$ (so-called "future null infinity"), does not intersect B. What's more, $B = M \setminus J^-(\mathcal{I}^+)$, and this captures that future-directed timelike curves in B can never reach future null infinity; this is the sense in which they are confined to a small region.

i.e., $\tilde{g} = \Omega^2 \phi_* g$

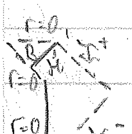
With this as a blueprint, then, the black hole region of a spacetime (M,g) can be defined for those spacetimes which admit a conformal compactification, i.e. a compact manifold with boundary $(\tilde{M},\tilde{g})$ s.t. there exists a conformal isometry $\phi: M \to \tilde{M}$ under which $\tilde{M} = \phi(M) \cup \partial\phi(M)$. In such a scenario, each complete null geodesic in M has an endpoint in $\partial\phi(M) \subseteq \tilde{M}$, and we may define $\mathscr{I}^+$ as before as the set of future endpoints. In this case, $B \subseteq M$ is defined as



$B := M \setminus J^-(\mathscr{I}^+)$ (now identifying M with $\phi(M)$) in the causal structure of $\tilde{M}$. As before, then, the black hole region is exactly the set of points in M from which an observer cannot escape to future null infinity.



What if singular behavior occurs outside of B, say if not every inextendible b-incomplete future-directed curve eventually remains B? This would imply that observers which get arbitrarily close to singular behavior can escape to infinity. If general relativity is to be a meaningful theory of gravity, we must be able to isolate singular behavior so as to say that a more refined (i.e. quantum) theory applies in the isolated regions but GR works well elsewhere. If singular behavior occurs outside of B (i.e. there are "naked singularities"), then no such isolation is possible, as information about singular behavior propogates out to infinity!



For this reason, Penrose suggested the so-called weak cosmic censorship conjecture, which claims

Generically, physically reasonable initial data for the Einstein equation does not develop naked singularities.

There exist formulations that do not invoke the structure of a conformal compactification. This conjecture is seen as one of the most important open problems in GR. Though some progress has been made in sperical symmetry for particular matter fields (Christodoulou, Dafermos) and for perturbations of Minkowski initial data (Christodoulou, Klainerman), the general statement has proved elusive for half a century.