Dirac Talks: Interpreting Coordinates in GR

James Wheeler

September 16, 2022

1 Day 1

In General Relativity, the universe is modeled as a 4-dimensional smooth *manifold*. We recall the definition of this latter term.

Definition 1. *A (Hausdorff, second countable) topological space M is said to be an n-dimensional manifold if every point $p \in M$ has a neighborhood that "looks like $\mathbb{R}^n$", i.e. there exist open sets $U \subset M$, $V \subset \mathbb{R}^n$, with $p \in U$, and a homeomorphism $\phi : U \to V$. The triple (U, V, ϕ) will be called a coordinate chart. Translations between overlapping coordinate charts, called transition functions, are required to be sufficiently smooth.*

This does not seem too unreasonable a model for reality: it would very much seem from our everyday intuition that we may put coordinates on our immediate surroundings in the set of spacetime "events". We may readily envision doing so, say, via the tools of a stopwatch and meter stick. We will return to inspect this intuition more closely later. There is much more one can say about further properties and structure of this manifold, but we will not dwell on such details.

An indispensable mathematical structure we associate to this manifold representing spacetime, however, is the *metric*. This is a means of taking inner products between, or assigning "lengths" to, tangent vectors on the manifold. The metric encodes all gravitational effects in General Relativity. This object is a generalization of special relativity's *spacetime interval*

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{1}$$

an intertial-frame-invariant measure of "separation" between two spacetime events. In General Relativity, the events must be taken to be only infinitesimally displaced in order to make sense of the separation (hence the metric's being defined on tangent vectors), but otherwise it measures the same physical quantity as the special relativistic analogue. That is, it is used to measure the proper time elapsed along a timelike worldline or the physical length along a spacelike curve, and to identify those directions which are lightlike. As an example, a metric we will be interested in is among the simplest nontrivial metrics of physical interest, the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2(\theta) d\varphi^2\right). \tag{2}$$

For our purposes, this will be defined for coordinate values satisfying $r > r_s, t \in \mathbb{R}, \theta \in (0, \pi)$, and $\varphi \in (0, 2\pi)$. In fact, we will primarily be concerned with radial phenomena, so we effectively only care about the set $\{(t, r) \in \mathbb{R}^2 \mid r > r_s\}$.

It is important to observe when parsing the above discussion that while the spacetime manifold is guaranteed, by definition, to admit *some* coordinates around any point, it does not immediately stipulate that these coordinates, in their numerical values, directly contain any physical content.

Any given manifold admits inordinately many coordinate charts, each with its own presentation of the metric. Indeed, it is almost universally the case that when one writes down a spacetime manifold of interest, the first or most natural choice of coordinates is not at all physically adapted and cannot be taken to have direct physical content. It is for this quite well-motivated reason that most computations center around quantities which are coordinate-invariant, so there is no question as to whether our particular choice of coordinates has biased us in an unphysical manner. Here, however, we wish to confront the issue of coordinates rather than avoid it.

Example 1. *Consider a spacetime described by a global coordinate chart with* $V = \{(t,x) \in \mathbb{R}^2 \mid x > 1\}$ *in which the metric takes the form*

$$ds^2 = -dt^2 + \frac{1}{(x-1)^2}dx^2.$$

In the new global coordinate chart with $\widetilde{V} = \{(t,y) \in \mathbb{R}^2\} = \mathbb{R}^2$ obtained by setting $y = \ln(x-1)$, we have

$$ds^2 = -dt^2 + dy^2.$$

Apparently, then, this spacetime is 2-dimensional Minkowski space, i.e. this is special relativity. Noticing that $y \to -\infty$ as $x \to 1$, this example demonstrates that even extreme coordinate properties, like being "finite" versus "infinite", need not meaningful even within special relativity.

□

While one can go through any number of manipulations and computations in any number of coordinate systems and learn a great deal about the spacetime at hand, the question which arises in our confrontation is the following: how are these grand manipulations tied to our humble beginnings of imagining building coordinates around a point with a stopwatch and meter stick? **When we compute test particle trajectories as geodesics in some coordinate system, how can these trajectories be concretely related back to our intuitive notions of what we see as everyday observers, in particular how we apparently assign times and distances to our immediate surroundings?**

Perhaps the most immediate way of attempting to characterize such "everyday coordinates" is that they should be approximately special relativistic. That is, we expect that they should have the property that the metric takes on a form very close to (1). This is given some theoretical weight by the following result of Semi-Riemannian geometry which guarantees that one can always find some such coordinates.

Proposition 1. *Let* (M,g) *be a Lorentzian manifold, and take* $p \in M$*. Then there exists a coordinate chart* (U,V,ϕ) *around* p *with the property that the metric takes the form (1) at* p*, with vanishing first partial derivatives at* p*. Further, free-fall motion through* p *is realized by straight lines through* $\phi(p)$ *in* $V \subset \mathbb{R}^4$*.*

The coordinates guaranteed here are called *normal coordinates.* At the very least, this proposition tells us that the model of GR is compatible with the hope that our everyday coordinates can be characterized as being approximately special relativistic, as we know that we can locally always find approximately special relativistic coordinates. That being said, it becomes apparent in considering the implications of the final sentence of this proposition that normal coordinates themselves don't fit the bill in general. This is because the final sentence indicates that free-fall motion is described as having constant velocity in such coordinates, which we know is not the case in the everyday coordinates of, say, observers on the surface of the Earth, according to whom their "natural" coordinates describe falling objects as accelerating.

Perhaps some other nearly special relativistic coordinates do fit the bill, however. To further understand our question, we now present a computation indicating that such coordinates can, in general, admit rather counterintuitive behavior, so we should be careful in taking this as a meaningful or complete constraint.

Example 2. *Schwarzschild Repulsion.*

We consider the Schwarzschild metric (2), and wish to understand the trajectories of test particles in radial freefall. Let $\gamma : I \to V$ be the coordinate expression of this trajectory, parameterized by proper time, so that $\gamma(\tau) = (t(\tau), r(\tau))$. To avoid going to the trouble of invoking the geodesic equation, we utilize a useful feature of the Schwarzschild geometry: the t-direction tangent vector field $\frac{\partial}{\partial t}$ is a Killing vector field (this is the observation that the metric components in (2) are all t-independent). This ensures that geodesic motion has the property that the quantity $E := -\langle \gamma'(\tau), \frac{\partial}{\partial t} \rangle$ is constant. We have

$$\gamma'(\tau) = \frac{dt}{d\tau}\frac{\partial}{\partial t} + \frac{dr}{d\tau}\frac{\partial}{\partial r},$$

so using the metric we can compute

$$E = \left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau}.$$

Since γ is parameterized by proper time, we have

$$-1 = \langle \gamma'(\tau), \gamma'(\tau) \rangle = -\left(1 - \frac{r_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2.$$

Dividing this equation by the square of the previous equation, we obtain

$$\frac{1}{E^2} = \left(1 - \frac{r_s}{r}\right)^{-1} - \left(1 - \frac{r_s}{r}\right)^{-3}\left(\frac{dr}{dt}\right)^2. \tag{3}$$

Rearranging yields

$$\left(\frac{dr}{dt}\right)^2 = \left[1 - \frac{1}{E^2}\left(1 - \frac{r_s}{r}\right)\right]\left(1 - \frac{r_s}{r}\right)^2.$$

Differentiating with respect to t and simplifying, we've found

$$\frac{d^2r}{dt^2} = \frac{r_s}{r^2}\left(1 - \frac{r_s}{r}\right)\left[1 - \frac{3}{2E^2}\left(1 - \frac{r_s}{r}\right)\right]. \tag{4}$$

This is the *coordinate acceleration* of a test particle in freefall in standard Schwarzschild static coordinates. In general, as we have said, we don't expect such coordinate-based quantities to be hugely meaningful. However, we notice that as $r \to \infty$, the metric (2) approaches the special relativistic expression (1). Let us see, then, if (4) tends to something reasonably resembling what we might expect from our everyday notion of acceleration in this regime. Taking the limit as $r \to \infty$, so that our test particle is moving very far from the Schwarzschild source and in the nearly special relativistic regime, (4) becomes

$$\frac{d^2r}{dt^2} \approx \left(1 - \frac{3}{2E^2}\right)\frac{r_s}{r^2}.$$

Meanwhile, (3) in this limit reads (setting $v = \frac{dr}{dt}$ to be the coordinate velocity) $\frac{1}{E^2} \approx 1 - v^2$, so substituting this in above finally reads

$$\boxed{\frac{d^2r}{dt^2} \approx \left(1 - 3v^2\right) a_{\text{newton}}} \tag{5}$$

where we have identified the usual attractive acceleration of Newtonian gravity $a_{\text{newton}} = -\frac{r_s}{2r^2} = -\frac{M}{r^2}$ (in natural units). This yields the rather surprising observation that the coordinate acceleration of a distant test particle is apparently *repulsive* whenever $v > \frac{1}{\sqrt{3}}$, or larger than about 58% of the speed of light.

If one follows the same procedure for a non-radial trajectory (also using that $\frac{\partial}{\partial \phi}$ is Killing), one finds the general result that the vector coordinate acceleration of a test particle moving very far from a Schwarzschild source tends to

$$\boxed{\vec{a} \approx \left[-\left(1 - 3v_r^2 + 2v_\varphi^2\right)\hat{r} + 2v_r v_\varphi \hat{\varphi}\right] |a_{\text{newton}}|} \tag{6}$$

This expression can be used numerically to correctly compute, for example, the precession of Mercury's perihelion, or the deflection angle of light in weak-field lensing.

□

What do we take away from this computation, and where does this leave our identification of everyday coordinates? Is there a meaningful sense in which cosmic rays are being decelerated by Earth's gravity? We leave these questions to be pondered until next week.

2 Day 2

We revisit the closing computation above. An objection one might make to bothering with an expression like (5) is that the surprising behavior doesn't seem practically measurable: it requires highly relativistic radial speeds, so it is unclear how one could possibly purport to measure such kinematic accelerations. We'll apply a boost to bring the problem more directly to bear. Let's take (5), as a reflection of how we would interpret acceleration $\dot{v}$ in special relativity, super-duper seriously. Then we would imagine that we can find the velocity of our test particle as measured by some observer O moving radially at constant velocity with respect to our original frame (in which the Schwarzschild source mass M is at rest) via the SR velocity-addition formula.

Say M moves at velocity v_M according to O, so we have that the velocity $\tilde{v}$ of our test particle as measured by O is

$$\tilde{v} = \frac{v_M + v}{1 + v_M v} \tag{7}$$

Denoting by $\tilde{r}$ and $\tilde{t}$ the distance between M and the test particle and the time coordinate according to O, the usual Lorentz transformations indicate that $\tilde{r} = r\sqrt{1 - v_M^2}$ and $d\tilde{t} = \frac{1+v_M v}{\sqrt{1-v_M^2}} dt$. Using these together with (7), we find

$$\begin{aligned}\tilde{a} &= \frac{d\tilde{v}}{d\tilde{t}} = \frac{d\tilde{v}}{dv}\frac{dv}{dt}\frac{dt}{d\tilde{t}} \\ &= \frac{1-v_M^2}{(1+v_M v)^2}\cdot\frac{\sqrt{1-v_M^2}}{1+v_M v}\cdot(1-3v^2)a_{\text{newton}} \\ &= \frac{(1-v_M^2)^{3/2}(1-3v^2)}{(1+v_M v)^3}\left(\frac{-M}{r^2}\right) \\ &= \frac{(1-v_M^2)^{5/2}(1-3v^2)}{(1+v_M v)^3}\tilde{a}_{\text{newton}}\end{aligned}$$

In particular, let us solve for the acceleration as seen by O when it is instantaneously the rest frame of our test particle, i.e. when $v_M = -v$. Making this substitution, we find

$$\tilde{a} = \frac{1-3v^2}{\sqrt{1-v^2}}\tilde{a}_{\text{newton}}$$

This computation represents the acceleration we would expect to assign (assuming we take (5) super-duper seriously) to an initially-at-rest test particle if a large gravitational source were approaching it relativistically from very far outside its Schwarzschild radius. Interestingly, the repulsion sign remains, but we've picked up a factor of $1/\sqrt{1-v^2}$, which is *unbounded* as $v \to 1$! If we attempt to measure repulsion on an object we view as initially at rest, then, this seems to indicate that the effect can appear immensely stronger than the usual Newtonian attraction! There exists at least one paper on the arxiv suggesting we try to measure such a repulsion on a test mass affixed to the side of the LHC as relativistic particles zip by.

Perhaps the first hint, beyond intuitions against repulsion, that our general procedure should be taken with a heap of salt is that its result is not unique: the same computation done in isotropic coordinates, as opposed to the standard Schwarzschild static coordinates of (2), yields the conflicting expression (which also gives the correct perihelion precession and weak lensing deflection)

$$\vec{a}_{\text{isotropic}} \approx \left[-\left(1-3v_r^2+v_\varphi^2\right)\hat{r} + 4v_r v_\varphi \hat{\varphi}\right]|a_{\text{newton}}|.$$

This expression can be found, for example, in documentation published by NASA's Jet Propulsion Laboratory. While this agrees with (6) for radial motion, it disagrees when azimuthal components are added– it can even disagree as to when the test particle is being repelled versus attracted. If our "everyday coordinates" are to be well-defined, then, these expressions cannot both represent such coordinates' notion of acceleration.

Having seen the ambiguities and pathologies arising in attempting to characterize our everyday coordinates as "approximately special relativistic" alone, we now inspect our natural coordinates more closely. We began our discussion with the claim that we have an intuitive understanding of how we assign distances and times to our surroundings, perhaps by using the simple tools of a stopwatch and meter stick. While this is a reasonable enough idea, our pathologies are arising because we have not been entirely clear as to precisely how this assignment is done. There are two fundamental questions to be addressed:

(i) How do we ascribe simultaneity to events in our surroundings?

(ii) How do we ascribe distances to events in our surroundings?

Both of these must be answered with respect to our timelike wordline, the only place from which we are able to make definitive measurements.

There are at least two reasonable approaches to resolving these questions. One approach, Fermi coordinates, is very geometric in nature, involving emanating spacelike geodesics orthogonal to the worldline to establish local surfaces of simultaneity as well as distances to points within the surface. Were the geometry of spacetime all that's of interest to us, these coordinates would be a very natural choice. However, our objective here is to discern how that geometry, as it manifests itself in any coordinate system, is related to our immediate experience, and Fermi coordinates don't achieve that: we have no means of constructing spacelike geodesics from observations of our surroundings. Satisfactory answers must be constructible *both* geometrically *and* empirically with immediate on-worldline observations.

So, how do we resolve these questions empirically, with simple and natural observations only at events on our worldline? Along the worldline, the only kinematical parameter one can really observe directly is proper time (we may want to upgrade our stopwatch to an atomic clock), so this will be our fundamental quantity. We will answer the first question precisely as it is answered in special relativity. Indeed, the answer is described at the very beginning of Einstein's original publication on special relativity, in what's sometimes called the *Einstein synchronization condition.* The construction is as follows. To attribute a value of time to a nearby event $p \in M$, we arrange for a mirror to be present at p, emit a pulse of light from our worldline at proper time τ_- such that it reflects off the mirror at p, and we measure the proper time τ_+ at which we receive the pulse back on our worldline. The time coordinate of the event is then $t(p) = \frac{\tau_+ + \tau_-}{2}$.

We now turn to the question of distance. To ascribe values to distance, we should certainly consider how it is that we *define* the notion of distance. In doing so, we find that we have not been waving around meter sticks to answer (ii) since at least 1960. Indeed, in 1983, the SI meter was defined by the time-of-flight of light, setting the speed of light c in stone as a defined quantity (which we take to be 1 in our units). This indicates to us that our empirical procedure outlined above already determines what we should call the "distance" to the event p: $d(p) = \frac{\tau_+ - \tau_-}{2}$.

We have thus described a natural, direct means of empirically ascribing coordinates to the surroundings of an observer's worldline via proper time measurements and light signals. Such coordinates are often called *radar coordinates.* Is this approach geometrically viable, so that we can take computations done in a general coordinate system and translate them into radar coordinates, thereby tying abstract GR quantities and trajectories directly to empirical counterparts? The following proposition assures us that the answer is (locally) yes, and hence that the geometric structure of GR is consistent with our emergent picture of everyday coordinates.

Proposition 2. *Let (M, g) be a smooth, time-orientable Lorentzian manifold and $\gamma : I \subset \mathbb{R} \to M$ be a future-directed timelike curve. Fix an orthonormal frame field $\{\boldsymbol{e}_i\}_{i=1}^3$ along γ, i.e. smooth maps*

$$\boldsymbol{e}_i : I \to TM, \quad \boldsymbol{e}_i(s) \in T_{\gamma(s)}M, \quad \langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_{ij}, \quad \langle \boldsymbol{e}_i(s), \gamma'(s) \rangle = 0.$$

Then for each $s_0 \in I$ there exists a neighborhood of $p := \gamma(s_0)$ on which the Einstein synchronization condition with respect to γ yields a smooth coordinate chart.

A proof can be found in the appendix. We notice that this proposition, as one would expect, is *local* in nature. This leads us to a convincing argument that we really **must** describe the universe as a manifold in GR, i.e. that we cannot find a physically natural global coordinate chart: familiar observations of multiple-image lensing of distant cosmological objects indicates to us that radar coordinates **cannot** be extended globally in a bijective manner (setting aside the clear issues with the physical merit of radar coordinates at cosmological distances).

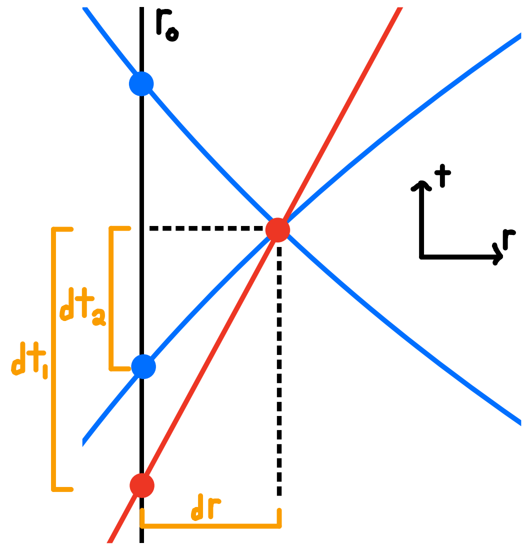

Figure 1: Constructing the radar coordinate velocity, as measured by a Schwarzschild static observe fixed at r_0 (the vertical black line). The plot is in standard coordinates. The red trajectory is the curve whose velocity we wish to measure, along which we identify two nearby points. The blue curves are outgoing and ingoing null curves (note the symmetry between them) which intersect the second point of interest on the red line. These are the trajectories of the light pulse used by the static observer at r_0 to characterize the distance and time of this point via the Einstein synchronization condition.

In closing, we now revisit the question of Schwarzschild repulsion, finding the acceleration of an object as measured by a static observer's radar coordinates. See the above figure.

Example 3. *Schwarzschild acceleration in radar coordinates.*

We compute the radar coordinate acceleration of a test particle as measured by a Schwarzschild static observer at $r_0 > r_s$. The coordinates r and t will refer to the usual Schwarzschild coordinates (with $v := \frac{dr}{dt}$), while r_e and t_e will refer to the radar coordinates (with $v_e := \frac{dr_e}{dt_e}$). We also abbreviate $\alpha := 1 - \frac{r_s}{r}$ and $\alpha_0 = 1 - \frac{r_s}{r_0}$. From the metric (2), the null curves must have coordinate velocity $\pm\alpha$– setting this condition, in the notation of the figure, for the outgoing null curve shown, we have

$$\alpha = \frac{dr}{dt_2}.$$

Meanwhile, we notice that $dr_e = \sqrt{\alpha_0}dt_2$ and $dt_e = \sqrt{\alpha_0}dt_1$ (we've used the symmetry between outgoing and incoming null curves) between the two identified red points. Using these and that $v = \frac{dr}{dt_1}$, we find

$$v_e = \frac{dr_e}{dt_e} = \frac{dt_2}{dt_1} = \frac{v}{\alpha}.$$

Though we've argued this at the intersection time, it is straightforward to generalize the image and see that this simple relation between the standard coordinate velocity and radar coordinate velocity holds in general. Notice that we've shown that the radar coordinate velocity here is just the ratio of the standard coordinate velocity to the standard coordinate speed of light, so this is measuring the fraction of the speed of light at which the test particle moves. Differentiating with respect to t_e, we have

$$\frac{dv_e}{dt_e} = \frac{dv_e}{dt}\frac{dt}{dt_e} = \frac{1}{\sqrt{\alpha_0}}\left[\frac{1}{\alpha}\frac{dv}{dt} - \frac{r_s}{r^2}\frac{v^2}{\alpha^2}\right] = \frac{1}{\sqrt{\alpha_0}}\left[\frac{1}{\alpha}\frac{dv}{dt} - \frac{r_s}{r^2}v_e^2\right] \tag{8}$$

To avoid taking any limits, we combine (3) and (4) from Example 2 to find

$$\frac{1}{\alpha}\frac{dv}{dt} = \left[1 - 3\frac{v^2}{\alpha^2}\right]\left(-\frac{r_s}{2r^2}\right) = (1 - 3v_e^2)\left(-\frac{r_s}{2r^2}\right).$$

Putting this into (8) yields

$$\boxed{a_{\text{radar}} = \frac{1 - v_e^2}{\sqrt{\alpha_0}}\left(-\frac{r_s}{2r^2}\right)}$$

Notice that *no limits have been taken in arriving at this expression*– it is true for any (radially moving) test particle in any static observer's radar coordinates, provided both are outside r_s. Further notice that we have avoided replacing $a_{\text{newton}} = -\frac{r_s}{2r^2}$, as this identification is really only meaningful in the $r \to \infty$ limit. One may get rid of the α_0 factor by taking the observer to infinity while still allowing for any test particle. In any event, the point of interest is that this quantity is always attractive, regardless of how quickly the test particle is moving and irrespective of any limits one might take. It seems somewhat miraculous that the correction term in (8) added on to our previously computed quantity $\frac{1}{\alpha}\frac{dv}{dt}$ is precisely what is needed to ensure a_{radar} has fixed sign. This computation demonstrates, then, that being precise about how we assign coordinates to our surroundings in an empirically meaningful manner resolves the issue of Schwarzschild repulsion. □

What we should take away from these considerations as a whole, and what I hope I've communicated in these talks, is that it can be easy to lose sight of the physical content of computations done in general relativity. Playing fast and loose is liable to lead one astray, so it is critical to keep oneself grounded by asking precise questions which have direct empirical content.

A Radar Coordinate Existence Proof

Here we present the core of the proof of Proposition 2, restated below. We draw several times upon standard results in Semi-Riemannian geometry found in Barrett O'Neill's text on the subject.

Proposition 2. *Let (M, g) be a smooth, time-orientable Lorentzian manifold and $\gamma : I \subset \mathbb{R} \to M$ be a future-directed timelike curve. Fix an orthonormal frame field $\{\boldsymbol{e}_i\}_{i=1}^3$ along γ, i.e. smooth maps*

$$\boldsymbol{e}_i : I \to TM, \quad \boldsymbol{e}_i(s) \in T_{\gamma(s)}M, \quad \langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \delta_{ij}, \quad \langle \boldsymbol{e}_i(s), \gamma'(s)\rangle = 0.$$

Then for each $s_0 \in I$ there exists a neighborhood of $p := \gamma(s_0)$ on which the Einstein synchronization condition with respect to γ yields a smooth coordinate chart.

Proof. WLOG, assume γ is parameterized by proper time τ. Fix $\tau_0 \in I$, let $O \subset M$ be a convex normal neighborhood of $p := \gamma(\tau_0)$, and take $\tau_\pm \in I$ such that $\tau_- < \tau_0 < \tau_+$ and $\Gamma := \gamma((\tau_-, \tau_+)) \subset O$. We now restrict all operations to occuring within the spacetime (O, g). Call $p_\pm := \gamma(\tau_\pm)$ and take

$$U := I_-(p_+) \cap I_+(p_-).$$

Then clearly $\forall \tau \in (\tau_-, \tau_+)$ we have $\gamma(\tau) \in U$ by definition, so $U \supset \Gamma$ is a nonempty open neighborhood of p. Fix $q \in U$ and note $p_\pm \in I_\pm(q)$, so in particular Γ intersects both of the disjoint open sets $I_\pm(q)$ (disjoint by Lemma 5.33 of O'Neil), and hence Γ intersects each of their boundaries $\partial I_\pm(q)$ by connectedness. As $I_\pm(q)$ are, respectively, future and past sets, $\partial I_\pm(q)$ are achronal by Corollary 14.27 of O'Neil, and so their intersections with Γ are unique. That is, there exist unique, well-defined maps

$$\alpha_\pm : U \to (\tau_-, \tau_+)$$

such that $\gamma(\alpha_\pm(q)) \in \partial I_\pm(q)$. By Lemma 14.2 of O'Neil, then, $\gamma(\alpha_\pm(q))$ are the unique points in Γ reachable from q via future- and past-directed null geodesics, and hence the Einstein synchronization condition uniquely determines a radial r_e and time t_e coordinate associated to each $q \in U$ given by

$$\begin{aligned} r_e(q) &= \frac{1}{2}\left[\alpha_+(q) - \alpha_-(q)\right] \\ t_e(q) &= \frac{1}{2}\left[\alpha_+(q) + \alpha_-(q)\right]. \end{aligned}$$

Take $\beta_q : [0,1] \to U$ to be the unique (since O is convex) null geodesic from $\beta_q(0) = q$ to $\beta_q(1) = \gamma(\alpha_+(q))$. Denoting by $n_q \in T_{\gamma(\alpha_+(q))}O$ the spacelike unit vector in the direction of the projection of $\beta_q'(1)$ orthogonal to $\gamma'(\alpha_+(q))$, we may further define spherical coordinates $\theta_e \in [0, \pi]$, $\phi_e \in (-\pi, \pi]$ on U via

$$\begin{aligned} \cos(\theta_e(q)) &= \langle \boldsymbol{e}_3(\alpha_+(q)), n_q \rangle \\ \sin(\theta_e(q))\sin(\phi_e(q)) &= \langle \boldsymbol{e}_2(\alpha_+(q)), n_q \rangle \\ \sin(\theta_e(q))\cos(\phi_e(q)) &= \langle \boldsymbol{e}_1(\alpha_+(q)), n_q \rangle. \end{aligned}$$

Clearly if $r_e(q_1) = r_e(q_2)$ and $t_e(q_1) = t_e(q_2)$, then $\alpha_\pm(q_1) = \alpha_\pm(q_2)$, so to check these coordinates are injective we must check that $\theta_e(q_1) = \theta_e(q_2)$, $\phi_e(q_1) = \phi_e(q_2)$ further implies that $q_1 = q_2$. These equalities ensure $n_{q_1} = n_{q_2}$, and hence that β_{q_1}, β_{q_2} describe the same null geodesic by geodesic uniqueness. Assume under these conditions that $q_1 \neq q_2$, so one of β_{q_1}, β_{q_2} contains both q_1, q_2, say WLOG that $\beta_{q_1}(s_0) = q_2$ for some $s_0 \in [0,1]$. Then Corollary 14.5 of O'Neil implies that the causal curve obtained by traversing the null geodesic from $\gamma(\alpha_-(q_1))$ to q_1 and then traversing β_{q_1} from q_1 to q_2 is itself a null geodesic, and hence that $\gamma(\alpha_+(q_1))$ is reachable from $\gamma(\alpha_-(q_1))$ via a null geodesic, contradicting lemma 14.2(1) of O'Neil since clearly $\gamma(\alpha_+(q_1)) \in I_+(\gamma(\alpha_-(q_1)))$ (as γ is timelike). Thus we have shown the coordinates $(t_e, r_e, \theta_e, \phi_e)$ are well-defined and injective.

It remains to show these coordinates are smooth. Define a map $U \times (\tau_-, \tau_+) \to TM$ into the tangent bundle by

$$(q, s) \mapsto v_q(s) := \Delta(q, \gamma(s)) = \exp_q^{-1}(\gamma(s)),$$

which is smooth in the input (q, s) by Lemma 5.9 of O'Neil, and set $h_q(s) := \langle v_q(s), v_q(s) \rangle$. Then $h_q(\alpha_\pm(q)) = 0$ since the geodesics from q to $\alpha_\pm(q)$ are null, and further

$$h_q'(s) = 2\langle v_q(s), (d\exp_q)^{-1}(\gamma'(s)) \rangle = 2\langle (d\exp_q)(v_q(s)), \gamma'(s) \rangle$$

by the Gauss Lemma, so $h_q'(\alpha_\pm(q)) \neq 0$ whenever $q \notin \Gamma$ as $(d\exp_q)(v_q(\alpha_\pm(q)))$ are then both nonzero null vectors (which cannot be orthogonal to the timelike vectors $\gamma'(\alpha_\pm(q))$). The implicit function theorem applied to the zero level set of $(q, s) \mapsto h_q(s)$ at each $(q, \alpha_\pm(q))$ therefore ensures that each of $\alpha_\pm$ is smooth on $U \backslash \Gamma$.

□