Math 216 Lecture Notes

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Day 1: Introduction and terminology (1.1)

Intro/Logistics (**see Canvas syllabus**):

- About me
- Course structure
- Grading policy
- Homework/deadlines/conflicts
- Prerequisites
- Office Hours: Monday, 5-7PM (EH B737) & Tuesday 2-3PM (1832 EH) & Friday 2-3PM (Math Lab)

Differential equations are fundamental to the modeling of physical phenomena.

Definition: A differential equation is an equation containing derivatives of one or more unknown functions (dependent variables) with respect to one or more independent variables.

Some famous differential equations fundamental to physics[1]:

(1) Newton's second law : single particle moving (in 1 dimension) in a force field.

$$\begin{aligned} F = ma &\iff F(x) = m\frac{d^2x}{dt^2} \\ &\iff F(x) = mx'' \end{aligned}$$

(2) Schrodinger's equation : single particle quantum mechanics.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi$$

(3) Maxwell's equations : electricity and magnetism.

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

(4) Einstein's equation : general relativity.

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

[1]These are not equations you need to know, nor necessarily even be able to parse. They simply illustrate the foundational role played by DE's in the sciences, as well as the utility of the terminology we're introducing to categorizing equations of physical interest.

These all contain parameters, variables that are neither the dependent nor independent variables.

DE's can be classified according to:

- Number of independent variables.

  Single independent variable $\implies$ ordinary differential equation (ODE).

  Multiple independent variables $\implies$ partial differential equation (PDE).

  We will care almost exclusively about ODE's.

- Dimension: number of dependent variables (unknown functions).

  Multiple unknowns $\implies$ multiple equations $\implies$ system of DE's.

- Order: the highest order derivative present in the equations.

- Structure: a DE is linear if the dependent variables and their derivatives only appear in a linear combination. E.g., a first-order linear ODE has the form $a(t)x' + b(t)x = c(t)$.

  Much of this semester will be spent studying linear ODE's, as they lend themselves to certain solution techniques.

Above,
(1) is a second-order ODE. It can be linear or nonlinear depending on the force function $F(x)$.
(2) is a second-order linear PDE.
(3) is a first-order system of linear PDE's.
(4) is a second-order system of nonlinear PDE's.

What does it mean to solve a differential equation?

Example: Consider the equation $x^4 + 3x - 4 = 0$. Is $x = -1$ a solution? Is $x = 1$ a solution? How do we know?

We check by plugging in the proposed solution:
$(-1)^4 + 3 \cdot (-1) - 4 = -6 \neq 0$, so setting $x = -1$ yields a **false** statement $\implies$ **not** a solution.
$1^4 + 3 \cdot 1 - 4 = 1 + 3 - 4 = 0$, so setting $x = 1$ yields a **true** statement $\implies$ **is** a solution.

□

Similarly, a proposed expression for the unknown function(s) solves a (system of) differential equation(s) if substituting it in yields a true statement.

Example: Which of the following are solutions to the ODE $x' = 2 - x$?

$x(t)$	Solution?	Initial condition
$x_1(t) = 2 + e^{-t}$	Yes	$x(0) = 3$
$x_2(t) = 2$	Yes	$x(0) = 2$
$x_3(t) = 2 + e^{t}$	No	N/A
$x_4(t) = 2 + Ce^{-t}$	Yes	$x(0) = 2 + C$

E.g., $x_1'(t) = -e^{-t}$, so

$$-e^{-t} = x_1'(t) \stackrel{?}{=} 2 - x_1(t) = 2 - (2 + e^{-t}) = -e^{-t}$$

is a true statement.

□

A DE will generally have infinitely many solutions. Solutions may be made unique by stipulating an initial condition, e.g. requiring $x(t_0) = x_0$ for a first-order equation. A DE together with an initial condition is an initial value problem (IVP).

Example: What is the solution to the IVP

$$x' = 2 - x, \qquad x(1) = 3?$$

We know $x(t) = 2 + Ce^{-t}$ is always a solution, and setting $3 = x(1) = 2 + Ce^{-1}$ yields $C = e$, so $x(t) = 2 + e \cdot e^{-t} = \boxed{2 + e^{1-t}}$

□

Generally, checking whether or when a proposed expression solves an ODE is straightforward:

Example: For which values of the parameters a and b does $x(t) = \sqrt{at + b}$ solve the IVP

$$x' = \frac{1}{x}, \qquad x(1) = 1?$$

For the proposed form of $x(t)$, we compute $x'(t) = \frac{a}{2\sqrt{at+b}}$, so the statement

$$\frac{a}{2\sqrt{at + b}} = x'(t) \stackrel{?}{=} \frac{1}{x(t)} = \frac{1}{\sqrt{at + b}}$$

is true so long as $a/2 = 1$, or $\boxed{a = 2}$. The initial condition then demands

$$1 = x(1) = \sqrt{a + b} = \sqrt{2 + b},$$

so we must have $\boxed{b = -1}$.

□

Day 2: Mathematical modeling via ODE's (1.1,2.1,4.1)

How can we describe practical problems via differential equations?

DE's are of interest due to their broad utility in quantitatively describing real-world phenomena.

The scenarios describable through differential equations are innumerable. They are used to model

- Heat flow
- Projectile trajectories
- Population growth and decay
- The structure of atoms
- Stock value variations
- Spread of diseases
- Satellite orbits
- Current flow in electrical circuits

and much, much more.

Among the simplest applications of DE's are those containing proportional growth or decay.

Example: A colony of rabbits reproduces at a rate proportional to its current population.

If $x(t)$ is the rabbit population as a function of time, it apparently satisfies the ODE

$$x' = rx.$$

for some constant $r > 0$ (the growth rate). One might observe that $x(t) = e^{rt}$ satisfies this ODE. To deduce the form of solutions systematically, note that $x' = rx \iff \frac{x'}{x} = r$, so that

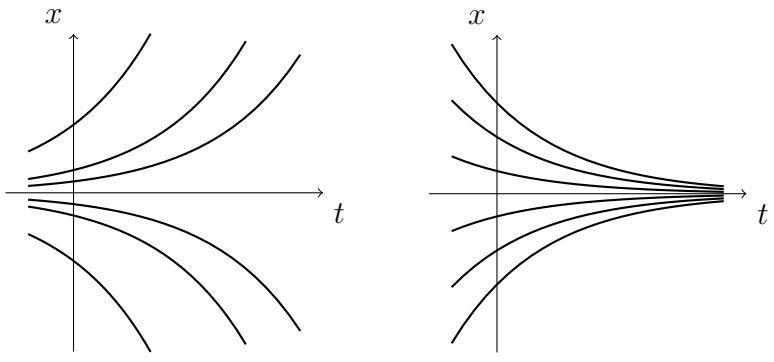
$$\begin{aligned} \frac{d}{dt}\left[\ln(x(t))\right] &= \frac{x'}{x} = r \\ \implies \qquad \ln(x(t)) &= \int r dt = rt + C \\ \implies \qquad x(t) &= e^{rt+C} = \tilde{C}e^{rt}. \end{aligned}$$

If the initial population is $x(0) = x_0$, then we must have $\tilde{C} = x_0$, so that $x(t) = x_0 e^{rt}$.

This is the basic reason that population growth is often considered to be exponential.

□

Similar behavior is ubiquitous, e.g. the decay of radioactive material ($r \to -r$ becomes a decay rate).



Simple growth/decay models for populations are limited. One improvement: species interactions.

Example: If $x(t)$ is a prey population and $y(t)$ is a predator population (e.g. rabbits and wolves), a basic model for their interactions is the Lotka-Volterra system

$$\begin{aligned} x' &= (a - \alpha y)x, \\ y' &= (-c + \gamma x)y, \end{aligned}$$

where a, α, c, and γ are positive constants. Two unknowns requires a system of two ODE's.

□

Many practical problems lend themselves to setting up a DE from their description.

Example: After many years of pollution, a 600,000 gallon pond has 3,000 kg of a particular pollutant. A proposed treatment project expects to be able to treat 1,000 gallons of polluted water per day, returning the water to the pond with a concentration of 1 g/gal. If the target concentration of the pollutant is 2 g/gal, how long will the treatment process take?

The primary quantity of interest is the mass (in kg) of the pollutant $P(t)$ present in the pond after t days. This setup lends itself to describing the rate of change $P'(t)$:

$$P'(t) = (\text{rate in}) - (\text{rate out}).$$

The "rate in" is the rate at which pollutant is added to the pond as purified water is returned:

$$\text{rate in} = \left(1{,}000\ \frac{\text{gal}}{\text{day}}\right) \cdot \left(1\ \frac{\text{g}}{\text{gal}}\right) = 1\ \frac{\text{kg}}{\text{day}}$$

The "rate out" is the rate at which pollutant is removed as polluted water is extracted:

$$\text{rate out} = \left(1{,}000\ \frac{\text{gal}}{\text{day}}\right) \cdot \left(\frac{P(t)}{600{,}000}\ \frac{\text{kg}}{\text{gal}}\right) = \frac{P(t)}{600}\ \frac{\text{kg}}{\text{day}}$$

These yield the IVP:

$$P'(t) = 1 - \frac{P(t)}{600}, \qquad P(0) = 3000$$

We would like to know the time t_* such that $P(t_*) = \left(2\ \frac{\text{g}}{\text{gal}}\right) \cdot (600{,}000\ \text{gal}) = 1{,}200\ \text{kg}$.

We'll be able to find t_* later, once we know how to construct the solution $P(t)$ to the IVP.

□

Newton's second law, $F = ma$, is a common source of simple ODE problems. Perhaps the most important example is a mass on a spring:

Example: A mass m on a spring of stiffness k, imparting a restoring force $F_s = -kx$ (Hooke's law) when stretched by x, is the prototypical simple harmonic oscillator. Newton's second law yields:

$$mx'' = -kx.$$

Practical systems are subject to damping, often proportional to velocity $F_d = -\gamma x'$, giving

$$mx'' + \gamma x' + kx = 0.$$

This is a second-order linear ODE for the position $x(t)$ of the mass.

It is often constructive to recast a second-order equation as a *system* of *first-order* equations. We can achieve this here by setting $y(t) := x'(t)$, so that the above indicates

$$my' + \gamma y + kx = 0.$$

This ODE for y is not self-contained since it includes the unknown $x(t)$; it is part of the system

$$\begin{aligned} x' &= y, \\ y' &= -\frac{k}{m}x - \frac{\gamma}{m}y. \end{aligned}$$

In ODE's, one can always trade order (derivatives) for dimension (additional unknowns) in this way.

We will study oscillator systems in great detail later on.

□

Day 3: Qualitative methods (1.1.3)

How can we deduce qualitative structure of solutions without a formula?

In general, solving DE's exactly is hard: there is no universal procedure to construct exact solutions.

Qualitative methods help understand solution behavior without having to find explicit formulas.

The most general first order ODE is

$$x' = f(t, x),$$

for some function $f(t, x)$ (sometimes called the rate function).

The value of $f(t, x)$ gives the rate of change of a solution passing through (t, x). In particular,

- $f(t, x) > 0 \implies x(t)$ is increasing.
- $f(t, x) < 0 \implies x(t)$ is decreasing.

Reflecting on the structure of $f(t, x)$ can provide a fair amount of information about solution curves.

Example: What is the solution to the IVP

$$x' = \sin(x), \qquad x(0) = \pi?$$

Setting $x(t) = \pi$, we see

$$0 = x'(t) \stackrel{?}{=} \sin(x(t)) = \sin(\pi) = 0$$

is a true statement, so this solves the DE. Clearly $x(0) = \pi$, so it also solves the IVP. It is **the** solution by uniqueness (seen later).

□

This was a precursor to the following:

Example: Consider the solution $x(t)$ to the IVP

$$x' = \sin(x), \qquad x(0) = 0.1.$$

Sketch $x(t)$ for $t > 0$: where is it increasing vs. decreasing? Concave up vs. concave down?

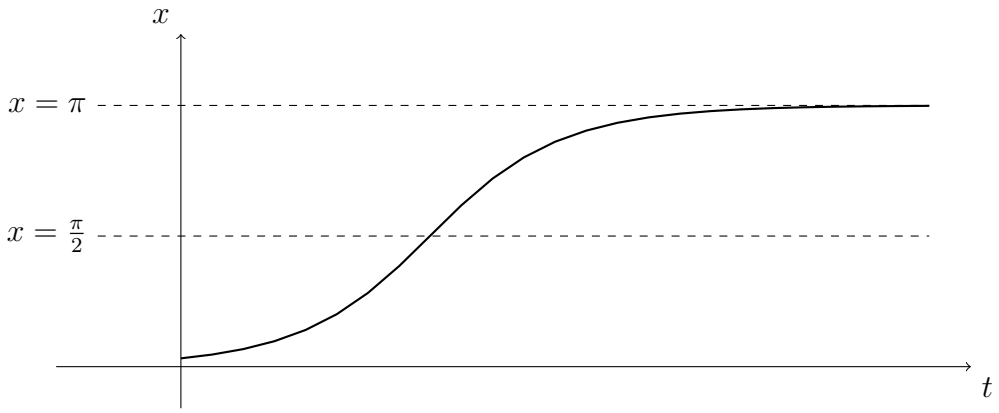
Since $x'(0) = \sin(x(0)) = \sin(0.1) > 0$, $x(t)$ initially increases. It will increase indefinitely unless it crosses $x = \pi$, where $x' = 0$. Since $x(t) = \pi$ is a solution to the DE, this crossing cannot happen.

$x(t)$ is therefore "trapped" between $x = 0.1$ and $x = \pi$ indefinitely for $t > 0$, always increasing.

For concavity, note that

$$x'' = \frac{d}{dt}[x'] = \frac{d}{dt}[\sin(x(t))] = \cos(x)x' = \sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

is positive ($x(t)$ is concave up) for $0 < x < \pi/2$ and negative ($x(t)$ is concave down) for $\pi/2 < x < \pi$.


x
$x = \pi$
$x = \frac{\pi}{2}$
t

□

Day 4: Qualitative methods (cont'd)

In the previous example, we saw how one might qualitatively describe a single solution.

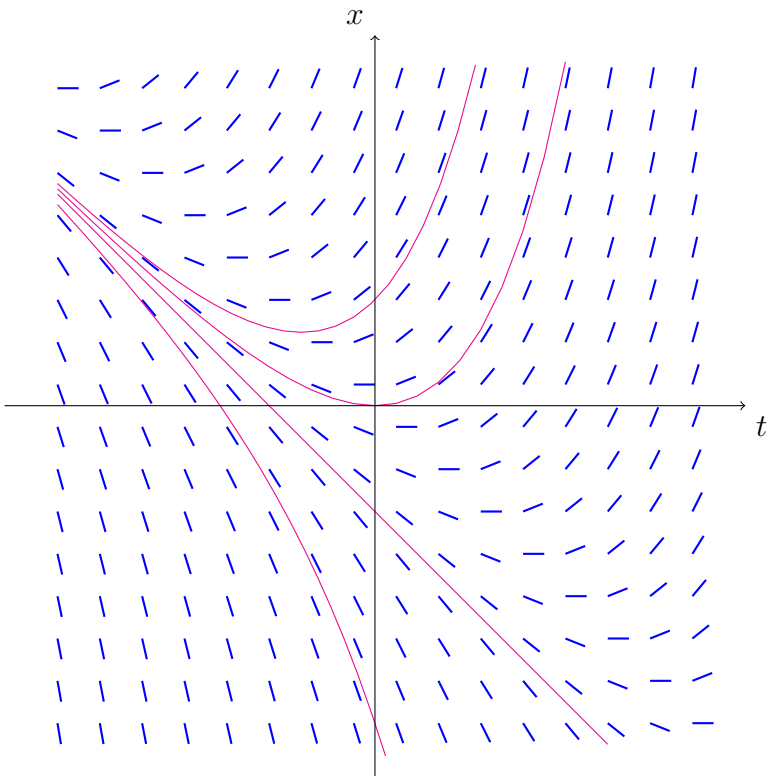
Going further, we can visualize the behavior of many solutions to a first-order ODE via a slope field.

The ODE $x' = f(t, x)$ determines the slope of solutions at each point (t, x) in the t-x plane; a slope field is a plot containing dashes with the slopes given by the ODE placed along a grid of (t, x) points.

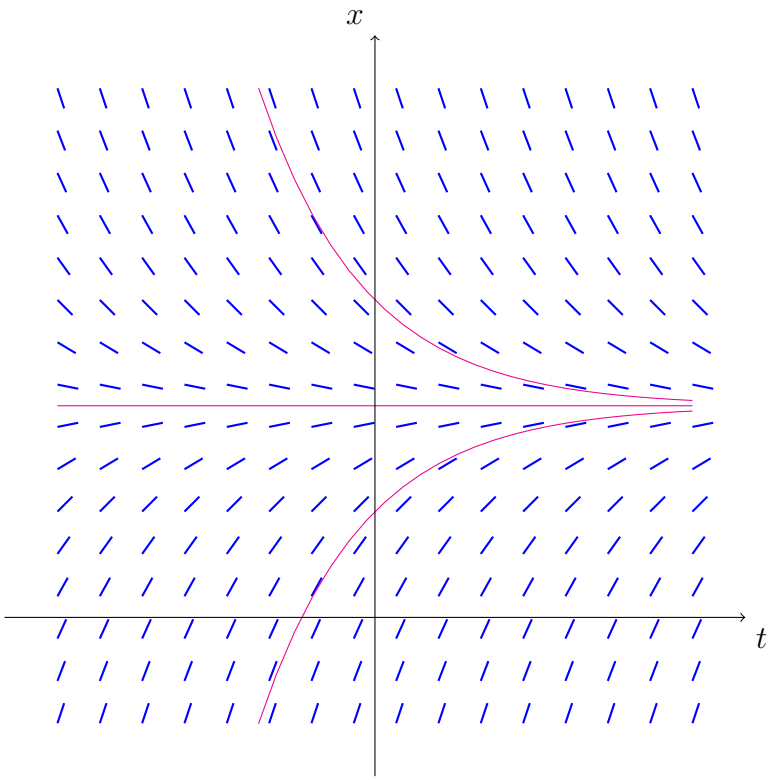
One could compute slopes by hand, but typically one only generates a slope field with a computer.

Example: Slope field for $x' = t + x$.

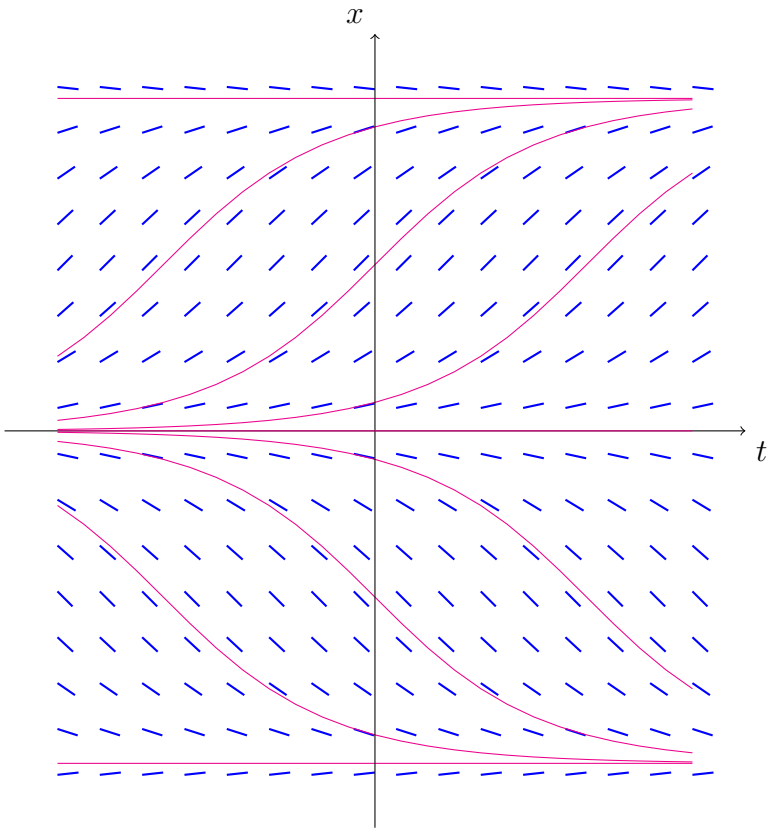
t	0	0	1	1	1	$\cdots$
x	0	1	0	1	-1	$\cdots$
x'	0	1	1	2	0	$\cdots$



Example: Slope field for $x' = 2 - x$.



Example: Slope field for $x' = \sin(x)$.



An important feature: the set of points at which $x' = 0$, where dashes are horizontal. This marks the boundary between solutions increasing versus decreasing.

This set is generally comprised of a collection of curves, called nullclines.

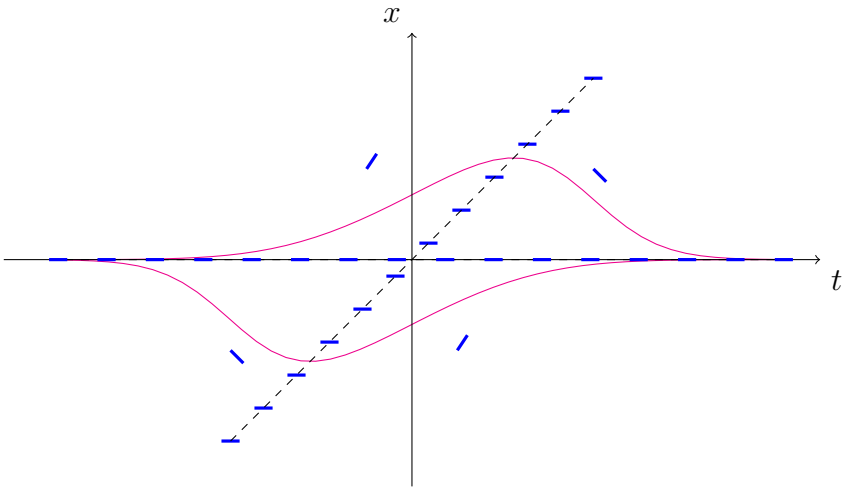
Nullclines can reveal solution structure without needing the entire slope field.

Example: Consider the differential equation

$$x' = (x - t)x.$$

Setting $x' = 0$, we see that the nullclines (dashed below) are the lines $x = 0$ and $x = t$.

These divide the t-x plane into regions where solutions are either all increasing or all decreasing:

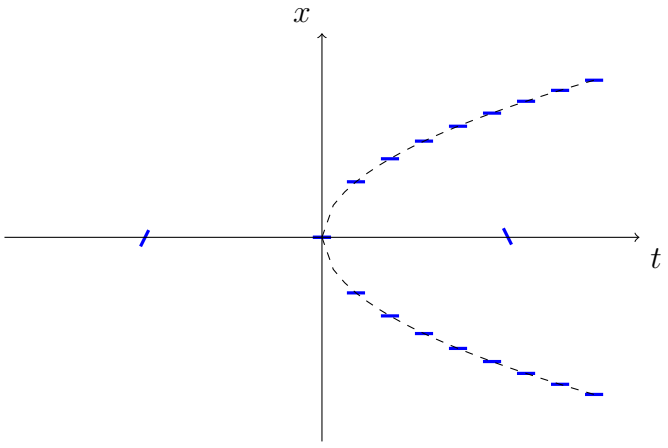


□

Example: Consider the differential equation

$$x' = x^2 - t.$$

Setting $x' = 0$, we see that the nullclines are the curves $x = \pm\sqrt{t}$.



A solution starting inside the parabola $x^2 = t$ will remain inside (for $t > t_0$) and decrease indefinitely.

□

Day 5: Euler's Method (6.2.1)

How can we quantify approximate solutions in practice?

Qualitative methods can provide a broad picture, but give little quantitative information.

In the great majority of endpoint applications, solutions are approximated numerically.

Consider the most general first order IVP,

$$x' = f(t, x), \qquad x(t_0) = x_0. \tag{1}$$

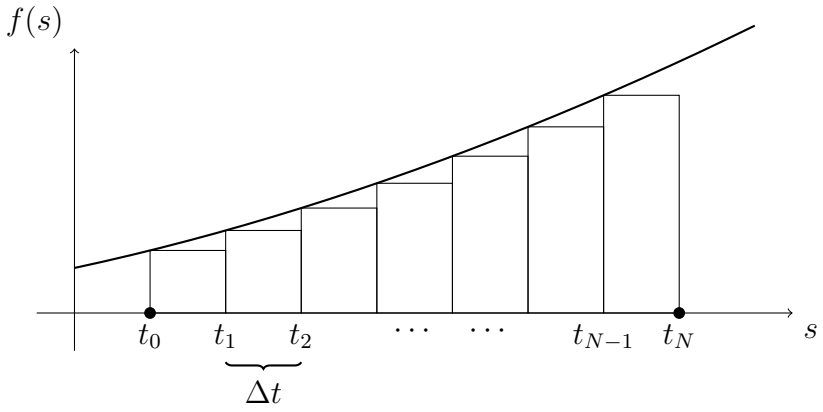
Simplest case: $f(t, x) = f(t)$, wherein the solution to $x'(t) = f(t)$ with $x(t_0) = x_0$ is

$$x(t) = x_0 + \int_{t_0}^{t} f(s)ds.$$

One can estimate $x(T)$ with a computer by approximating the integral via a Riemann sum.

Dividing (t_0, T) into N subintervals of width $\Delta t = \frac{T-t_0}{N}$ with endpoints, $t_0, t_1, \ldots, t_N$ $(t_N = T)$,

$$x(T) \approx x_0 + \sum_{i=0}^{N-1} f(t_i)\Delta t$$



The area of each rectangle contributes $f(t_i)\Delta t = x'(t_i)\Delta t$ to the value of $x(T)$, so the approximate change in x over each subinterval is its initial rate of change multiplied by the interval's width Δt.

Note the rectangles underestimate the integral when f is increasing, i.e. when $x(t)$ is concave up.

Conversely, they would overestimate when f is decreasing, i.e. when $x(t)$ is concave down.

Euler's method simply extends this approximation scheme to the general IVP (1):

Dividing (t_0, T) into N steps as above, the approximate value X_{n+1} of $x(t_{n+1})$ is obtained recursively by adding a change to X_n given by multiplying the rate of change $x'(t_n) \approx f(t_n, X_n)$ with Δt,

$$\boxed{X_{n+1} = X_n + f(t_n, X_n)\Delta t}$$

Starting with $X_0 = x_0$, we may recursively compute $X_1, X_2, \ldots, X_N$ to estimate $x(T) \approx X_N$.

Example: Consider the solution $x(t)$ to the IVP

$$x' = xt, \qquad x(1) = 2.$$

Use Euler's method with $N = 4$ steps to approximate $x(3)$.

Here $f(t, x) = xt$. We have $t_0 = 1$ and $T = 3$, so $\Delta t = \frac{T - t_0}{N} = \frac{1}{2}$. Starting with $X_0 = 2$,

$$\begin{aligned}
X_1 &= X_0 + f(t_0, X_0)\Delta t = 2 + (2 \cdot 1) \cdot \frac{1}{2} &&= 3, \\
X_2 &= X_1 + f(t_1, X_1)\Delta t = 3 + (3 \cdot 1.5) \cdot \frac{1}{2} &&= 21/4, \\
X_3 &= X_2 + f(t_2, X_2)\Delta t = \frac{21}{4} + \left(\frac{21}{4} \cdot 2\right) \cdot \frac{1}{2} &&= 21/2, \\
X_4 &= X_3 + f(t_3, X_3)\Delta t = \frac{21}{2} + \left(\frac{21}{2} \cdot 2.5\right) \cdot \frac{1}{2} &&= 189/8,
\end{aligned}$$

Hence $\boxed{x(3) \approx X_4 = 189/8 = 23.625}$. Some like to carry out this procedure in a table:

n	t_n	X_n	$f(t_n, X_n)$	ΔX
0	1	2	2	1
1	1.5	3	4.5	2.25
2	2	5.25	10.5	5.25
3	2.5	10.5	26.25	13.125
4	3	23.625	$\cdots$	$\cdots$

□

The discrete collection of approximate points (t_n, X_n) constitutes a numerical solution to the IVP.

Vary the k slider in this Desmos demo to see how a numerical solution improves with larger N.

In particular, for large N the error $|x(T) - X_N|$ is generally proportional to $1/N$.

Euler's method **undershoots** when $x(t)$ is concave up and **overshoots** when $x(t)$ is concave down.

A point of caution: Euler's method cannot definitively discern when a solution terminates.

Example: Consider the solution to the IVP

$$x' = x^2, \qquad x(0) = 1$$

Use Euler's method with $N = 3$ steps to estimate $x(1.5)$.

Here $f(t, x) = x^2$. We have $t_0 = 0$ and $T = 1.5$, so $\Delta t = \frac{T - t_0}{N} = 0.5$. Starting with $X_0 = 1$,

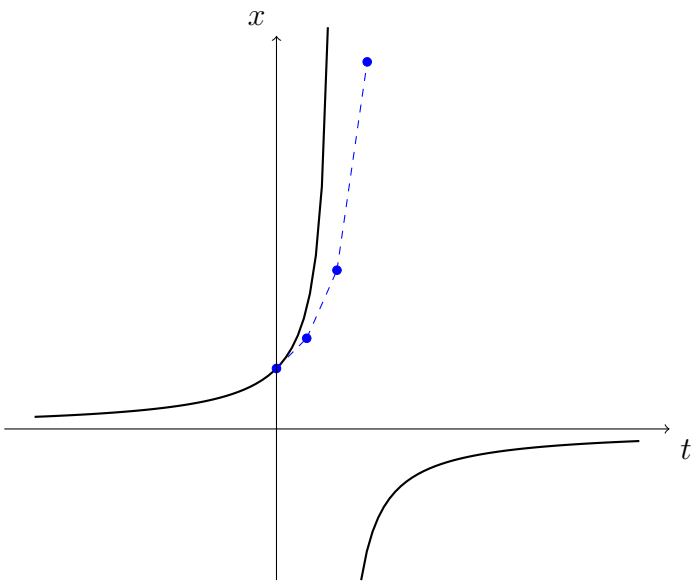
n	t_n	X_n	$f(t_n, X_n)$	ΔX
0	0	1	1	0.5
1	0.5	1.5	2.25	1.125
2	1	2.625	6.89	3.445
3	1.5	6.07	$\cdots$	$\cdots$

This procedure carried through entirely normally, giving the estimate $x(1.5) \approx X_3 \approx \boxed{6.07}$.

However, **this estimate is meaningless**.

The solution to this IVP turns out to be $x(t) = \frac{1}{1-t}$, which diverges at $t \to 1$. There is no way to continuously proceed along the graph of x from $t_0 = 0$ to $T = 1.5$, as Euler's method attempts.

We say $x(t) = \frac{1}{1-t}$ is a solution only on $(-\infty, 1)$, as this is where it is tied to the initial condition.



Takeaway: solutions may terminate in finite time, and numerics cannot detect this with certainty.

One should be skeptical of numerics which seem to blow up as one takes N larger and larger.

□

Day 6: Autonomous Equations (1.5)

Recall the most general first order ODE,

$$x' = f(t, x).$$

Such an ODE is called autonomous if $f(t, x) = f(x)$ depends only on x:

$$x' = f(x).$$

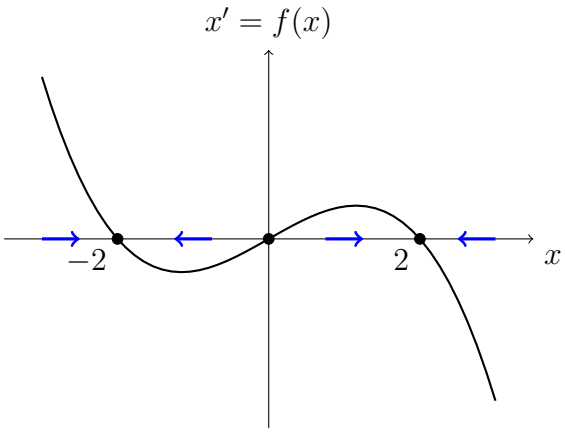
The sign of $f(x)$ again tells us solutions are increasing or decreasing when x takes a particular value:

- $f(x) > 0 \implies x(t)$ is increasing
- $f(x) < 0 \implies x(t)$ is decreasing
- $f(x) = 0 \implies x(t)$ is constant (values x_* of x such that $f(x) = 0$ are equilibrium solutions or critical points).

Encoded in the phase line, the x-axis adorned with arrows pointed toward solution progression:

Example: Phase line for $x' = (4 - x^2)x$.

Here $f(x) = (4-x^2)x$, and equilibria are $x_* = 0, \pm 2$. We plot $f(x)$ to help construct the phase line.



Arrows point towards $x_* = \pm 2 \implies$ nearby solutions limit to these values as $t \to \infty$. We call these stable equilibria (or critical points).

Arrows point away from $x_* = 0 \implies$ nearby solutions diverge away from this value as $t \to \infty$. This is an unstable equilibrium.

□

Example: Limited population growth.

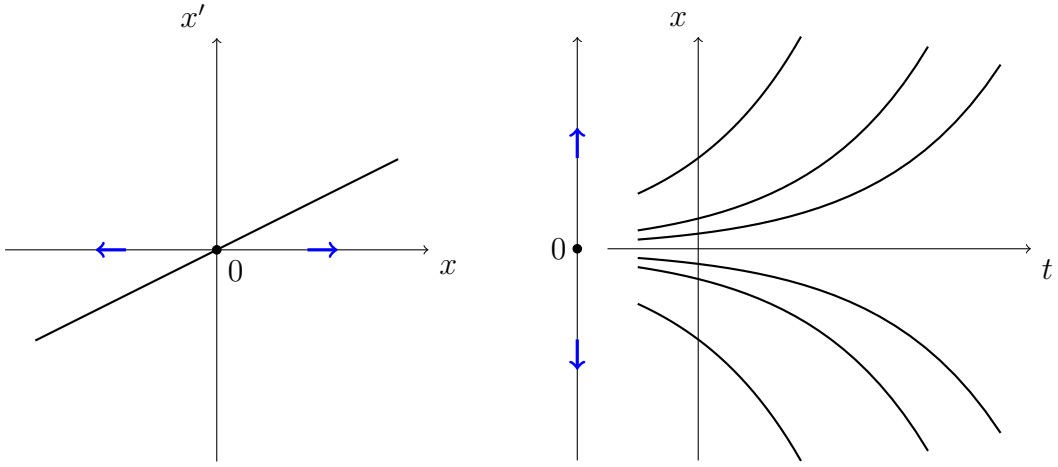
Recall that a basic model of population growth is the IVP

$$x' = rx, \qquad x(0) = x_0 \tag{2}$$

where $r > 0$ is the rate of growth. The solution is readily found to be exponential growth,

$$x(t) = x_0 e^{rt}.$$

The phase line and solution curves are shown below.



This is an accurate model for some processes (e.g. radioactive decay, fixed-interest appreciation).

Ecological populations are often subject to constraints, whereby the rate r varies: $r \to R(x)$, or

$$x' = R(x)x.$$

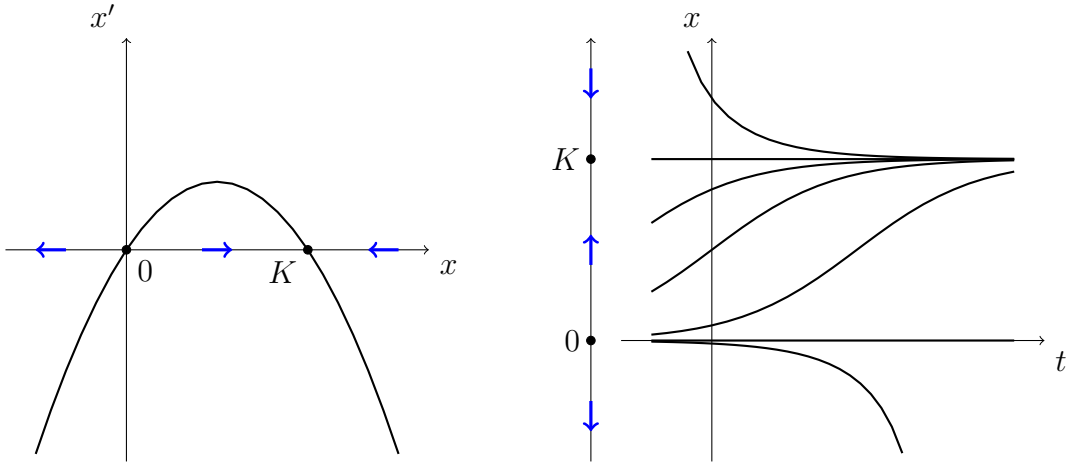
In particular, while (2) is a good model in certain regimes, growth is often limited to a maximum sustainable level $K > 0$ at which $R(K) \to 0$, called the carrying capacity.

This is captured simply by taking a linear effective growth rate, $R(x) = r(1 - \frac{x}{K})$, resulting in

$$x' = r\left(1 - \frac{x}{K}\right)x, \tag{3}$$

an ODE known as the logistic equation.

One can easily construct the phase line and deduce the qualitative structure of solutions:



The capacity K is a stable equilibrium that population limits to, whether initially larger or smaller. □

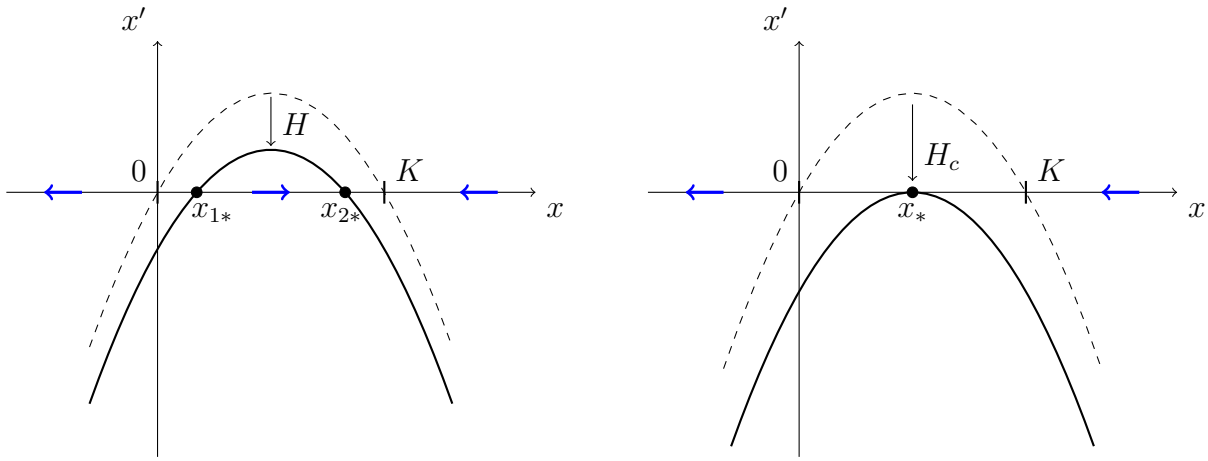
In animal husbandry or ecological management, one might be concerned with the impact of harvesting from a population (hunting, fishing, etc.).

This can be modeled by subtracting a constant harvesting rate H from x' in (3),

$$x' = r\left(1 - \frac{x}{K}\right)x - H. \tag{4}$$

The parabola in the x-x' plane is now shifted down by H, generally moving the equilibria inward.

There is a critical value H_c where the parabola just touches the axis and the equilibria coalesce.



Algebraically, the equilibria can be solved for by setting $x' = 0$ in (4):

$$x_* = \frac{K}{2}\left[1 \pm \sqrt{1 - \frac{4H}{rK}}\right],$$

from which we see that $H_c = \frac{rK}{4}$.

For $H < H_c$, harvesting is sustainable: the population settles at the stable equilibrium $x_{2*} < K$.

For $H > H_c$, harvesting is unsustainable: there is no equilibrium, so the population always dies out.

When $H = H_c$, arrows point toward x_* on one side and away on the other; x_* is semistable.

It is often useful to understand how the structure of equilibria changes with a parameter.

Day 7: Separation of variables (1.3)

How can we construct exact solutions?

There is no universal answer, but there are techniques for certain types of equations.

Recall the most general first order ODE,

$$x' = f(t, x).$$

We've seen that if $f(t, x) = f(t)$, the general solution is

$$x(t) = \int f(t)dt,$$

so that solving ODE's is closely related to integration.

A first-order ODE $x' = f(t, x)$ is called separable if $f(t, x) = p(t)q(x)$, i.e. if

$$x' = p(t)q(x).$$

Example: Which of the following correspond to separable equations?

$f(t, x)$	Separable?
$f(t, x) = t^2 x$	Yes
$f(t, x) = t + x$	No
$f(t, x) = e^{t+x}$	Yes
$f(t, x) = \ln(tx)$	No
$f(t, x) = t + x^2 t$	Yes

□

Separable equations can be solved via separation of variables:

$$\begin{aligned} x'(t) &= p(t)q(x) \\ \frac{x'(t)}{q(x)} &= p(t) \\ \int \frac{x'(t)}{q(x)} dt &= \int p(t)dt \\ \int \frac{dx}{q(x)} &= \int p(t)dt. \end{aligned}$$

The last step was a normal change of variables in the integral (a "u-substitution" using x instead of u). A convenient heuristic shorthand is to rewrite $x'(t) = \frac{dx}{dt}$ and "multiply" by dt:

$$\begin{aligned} \frac{dx}{dt} &= p(t)q(x) \\ \frac{dx}{q(x)} &= p(t)dt \\ \int \frac{dx}{q(x)} &= \int p(t)dt. \end{aligned}$$

Regardless, computing both integrals leads to an equation determining $x(t)$ (at least implicitly).

Example: Find all possible solutions to $x' = 2 - x$.

$$\begin{aligned} \frac{dx}{dt} &= 2 - x \\ \frac{dx}{2-x} &= dt \\ \int \frac{dx}{2-x} &= \int 1 dt \\ -\ln|2-x| &= t + C \\ 2 - x &= \pm e^{-t-C} = -\tilde{C}e^{-t} \\ \boxed{x(t) &= 2 + \tilde{C}e^{-t}} \end{aligned}$$

□

Example: We may use separation of variables to solve the logistic equation (3) IVP,

$$x' = r\left(1 - \frac{x}{K}\right)x, \qquad x(0) = x_0.$$

Using partial fraction decomposition to note that

$$\frac{1}{\left(1 - \frac{x}{K}\right)x} = \left[\frac{1}{(K-x)} + \frac{1}{x}\right],$$

we obtain via separation of variables

$$\begin{aligned} \int \frac{dx}{\left(1 - \frac{x}{K}\right)x} &= \int r dt \\ \ln\left|\frac{x}{K-x}\right| &= rt + C \\ \implies \quad \frac{x}{K-x} &= \tilde{C}e^{rt} \\ \implies \quad x(t) &= \frac{K}{1 + \tilde{C}e^{-rt}}. \end{aligned}$$

Requiring $x(0) = x_0$, we find

$$\boxed{x(t) = \frac{Kx_0}{x_0 + (K - x_0)e^{-rt}}} \tag{5}$$

Note that $\lim_{t\to\infty} x(t) = K$ (for $x_0 > 0$), in agreement with our earlier phase line analysis.

□

Day 8: Separation of variables (cont'd)

Example: Characterize all possible solutions to

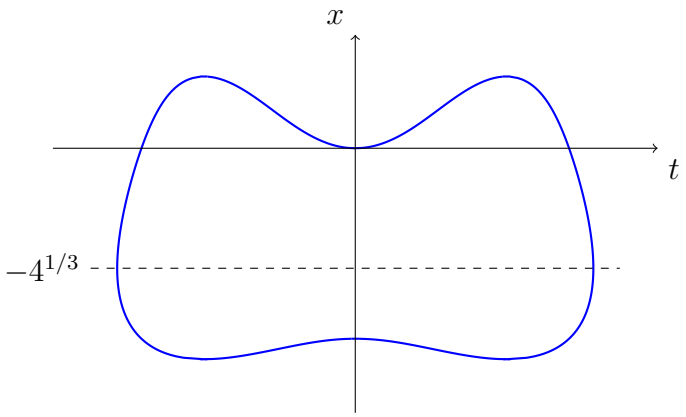
$$x' = \frac{4t - t^3}{4 + x^3}.$$

What is the interval of existence for the solution starting at $x(0) = 0$?

$$\begin{aligned}
\frac{dx}{dt} &= \frac{4t - t^3}{4 + x^3} \\
(4 + x^3)dx &= (4t - t^3)dt \\
\int (4 + x^3)dx &= \int (4t - t^3)dt \\
4x + x^4/4 &= 2t^2 - t^4/4 + C \\
\boxed{\tilde{C} = x^4 + 16x + t^4 - 8t^2}
\end{aligned}$$

This relation fully characterizes solutions $x(t)$, but only implicitly: the graph of $x(t)$ is a subset of the points (t, x) that satisfy the relation. These points can be plotted with many tools (e.g. Desmos).

For the solution with $x(0) = 0$, we have $\tilde{C} = 0$, giving the following plot:



The solution $x(t)$ apparently terminates at "turnaround points" where $x' \to \pm\infty$. We see from the ODE that this occurs at $x = -4^{1/3}$, where

$$0 = t^4 - 8t^2 - 3 \cdot 4^{4/3}.$$

This is a quadratic equation for t^2, with solutions t_* given by

$$\begin{aligned}
t_*^2 &= \frac{8 \pm \sqrt{64 + 12 \cdot 4^{4/3}}}{2} = 4 \pm \sqrt{16 + 3 \cdot 4^{4/3}}, \\
\implies t_* &= \pm\sqrt{4 + \sqrt{16 + 3 \cdot 4^{4/3}}} \approx \pm 3.15,
\end{aligned}$$

so this solution exists on the range $(-3.15, 3.15)$. Note we chose the first $\pm$ to be $+$ to only report the real solutions. These are indeed the left- and right-most t values in the plot above (see Desmos)

□

Recall that Newton's second law frames a variety of physical questions in terms of ODE's.

Example: A 10 kg package is dropped from rest with a parachute from an airplane at an altitude of 1 km. As the package falls, it experiences the forces of gravity (the acceleration due to gravity is $g \approx 10$ m/s^2) and air resistance on the parachute, proportional to the square of its velocity with proportionality constant $\gamma = 4$ kg/m. How long does it take the package to reach the ground?

If $x(t)$ is the distance the package has fallen (in meters) after t seconds, Newton's second law says

$$mx''(t) = F_g - F_d = mg - \gamma(x'(t))^2,$$

where $m = 10$ kg is the package's mass. This is a second-order equation.

Setting $v(t) := x'(t)$, this becomes a first-order equation for v,

$$\begin{aligned} v' &= g - \frac{\gamma}{m}v^2 \\ &= 10 - 0.4v^2. \end{aligned}$$

This DE is separable:

$$\int \frac{dv}{10 - 0.4v^2} = \int dt = t + C$$

The remaining integral requires partial fraction decomposition, namely

$$\begin{aligned} \frac{1}{10 - 0.4v^2} &= \frac{1}{4}\left[\frac{1}{5 - v} + \frac{1}{5 + v}\right] \\ \implies \int \frac{dv}{10 - 0.4v^2} &= \frac{1}{4}\ln\left|\frac{5 + v}{5 - v}\right| \end{aligned}$$

Hence separation of variables has yielded

$$\ln\left|\frac{5 + v}{5 - v}\right| = 4t + C,$$

and we may solve this for v to obtain

$$v(t) = 5\left(\frac{\tilde{C}e^{4t} - 1}{\tilde{C}e^{4t} + 1}\right)$$

Since the package was dropped from rest, $v(0) = 0$, yielding

$$\begin{aligned} 0 = v(0) &= 5\left(\frac{\tilde{C} - 1}{\tilde{C} + 1}\right) \\ \iff \quad \tilde{C} &= 1 \\ \implies v(t) &= 5\left(\frac{e^{4t} - 1}{e^{4t} + 1}\right) \end{aligned}$$

This tells us $v(t)$, but we're really interested in $x(t)$. Since $x'(t) = v(t)$, we simply integrate:

$$x(t) = \int v(t)dt = \frac{5}{2}\ln\left(e^{2t} + e^{-2t}\right) + C_2.$$

Requiring $x(0) = 0$ as well to obtain C_2, we find

$$x(t) = \frac{5}{2}\ln\left(\frac{e^{2t} + e^{-2t}}{2}\right).$$

Setting $x(t_*) = 1,000$, we can use a computer to find that $t_* \approx 200.347$ seconds.

Note that $v(t) \to 5$ m/s very quickly. This is the package's terminal velocity, and it is a stable equilibrium of the ODE for v. It is no coincidence, then, that t_* is very nearly $\frac{1{,}000\text{ m}}{5\text{ m/s}} = 200$ s.

□

Day 9: First-order linear ODE's (1.4)

How can we construct (more) exact solutions?

The standard form of a first-order linear ODE is

$$x' + p(t)x = g(t). \tag{6}$$

We call such an equation homogeneous if $g(t) = 0$ and nonhomogeneous otherwise.

Linear differential equations are of particular interest because they admit a superposition principle:

Theorem: If $x_1(t)$ and $x_2(t)$ are, respectively, solutions to

$$x' + p(t)x = g_1(t),$$
$$x' + p(t)x = g_2(t)$$

for some functions $p(t)$, $g_1(t)$, $g_2(t)$, then the linear combination $\tilde{x}(t) := c_1x_1(t) + c_2x_2(t)$ solves

$$x' + p(t)x = c_1g_1(t) + c_2g_2(t)$$

for any constants c_1, c_2.

Proof: We compute

$$\begin{aligned}\tilde{x}' + p(t)\tilde{x} &= (c_1x_1 + c_2x_2)' + p(t)(c_1x_1 + c_2x_2)\\ &= c_1x_1' + c_2x_2' + c_1p(t)x_1 + c_2p(t)x_2\\ &= c_1(x_1' + p(t)x_1) + c_2(x_2' + p(t)x_2)\\ &= c_1g_1 + c_2g_2.\end{aligned}$$

□

Key implications:

- If $x_1(t)$ and $x_2(t)$ solve a homogeneous DE $x' + p(t)x = 0$, then so does any such $\tilde{x}(t)$.
- If $g_1(t) = g_2(t) = g(t)$, then $x_1(t) - x_2(t)$ solves the homogeneous problem.

The latter observation implies: every solution to the linear ODE (6) takes the form

$$x_g(t) = x_h(t) + x_p(t),$$

where x_h is the general solution to the homogeneous problem and x_p is a single, particular solution to the nonhomogeneous problem. We will draw on these ideas extensively later on.

For now, we simply note that a key step in solving any linear DE is solving the homogeneous problem.

For first-order equations (6), solving the homogeneous problem is procedurally straightforward:

$$
\begin{aligned}
& x' + p(t)x = 0 \\
\implies \qquad & \frac{dx}{dt} = -p(t)x \\
& \int \frac{dx}{x} = -\int p(t)dt \\
& \ln|x| = -P(t) + C \\
\implies \qquad & x_h(t) = \tilde{C}e^{-P(t)}
\end{aligned}
$$

Here, $P(t)$ is simply some antiderivative of $p(t)$.

Armed with the homogeneous solution x_h, we solve the full problem via variation of parameters.

Idea: form an ansatz (or guess) by promoting the unknown constant $\tilde{C}$ in x_h to a function $A(t)$.

That is, we simply guess $x(t) = A(t)e^{-P(t)}$ and see where (6) takes us:

$$
\begin{aligned}
g(t) &= x' + p(t)x \\
&= \left[A(t)e^{-P(t)}\right]' + p(t)A(t)e^{-P(t)} \\
&= \left[A'(t)e^{-P(t)} - P'(t)A(t)e^{-P(t)}\right] + p(t)A(t)e^{-P(t)} \\
&= A'(t)e^{-P(t)} \\
\implies \qquad A'(t) &= g(t)e^{P(t)} \\
\implies \qquad A(t) &= \int \left[g(t)e^{P(t)}\right] dt
\end{aligned}
$$

Having found $A(t)$, we conclude that the general solution to the nonhomogeneous problem is

$$
\boxed{x(t) = e^{-P(t)} \int g(t)e^{P(t)}dt}
$$

The abstraction here is made much more concrete in an example.

Example: Solve the IVP

$$
(1+t)x' + 2x = 6t, \qquad x(1) = 1.
$$

Rather than memorize formulas, we recall the procedure. First, solve the homogeneous problem:

$$
\begin{aligned}
& (1+t)x' + 2x = 0 \\
\implies \; & (1+t)\frac{dx}{dt} = -2x \\
& \int \frac{dx}{x} = -\int \frac{2}{1+t}dt \\
& \ln|x| = -2\ln|1+t| + C \\
\implies \qquad & x_h(t) = \frac{\tilde{C}}{(1+t)^2}
\end{aligned}
$$

Promoting $\tilde{C} \to A(t)$, we now take the ansatz $x(t) = \frac{A(t)}{(1+t)^2}$ and plug into the original problem,

$$\begin{aligned}
6t &= (1+t)x' + 2x \\
&= (1+t)\left[\frac{A(t)}{(1+t)^2}\right]' + \frac{2A(t)}{(1+t)^2} \\
&= (1+t)\left[\frac{A'(t)}{(1+t)^2} - 2\frac{A(t)}{(1+t)^3}\right] + \frac{2A(t)}{(1+t)^2} \\
&= \frac{A'(t)}{1+t} \\
\implies \quad A'(t) &= 6t(1+t) \\
\implies \quad A(t) &= 2t^3 + 3t^2 + C,
\end{aligned}$$

so that the general solution to our ODE is

$$x(t) = \frac{2t^3 + 3t^2 + C}{(1+t)^2}.$$

Setting the initial condition, we find

$$\begin{aligned}
1 = x(1) &= \frac{5+C}{4} \\
\iff \quad C &= -1 \\
\implies \quad &\boxed{x(t) = \frac{2t^3 + 3t^2 - 1}{(1+t)^2} = 2t - 1}
\end{aligned}$$

□

Day 10: First-order linear ODE's (cont'd)

Example: Recall the pond pollutant problem from Day 2, wherein we wished to solve the IVP

$$P'(t) = 1 - \frac{P(t)}{600}, \qquad P(0) = 3000$$

and determine the time t_* such that $P(t_*) = \left(2 \frac{\text{g}}{\text{gal}}\right) \cdot (600{,}000 \text{ gal}) = 1{,}200 \text{ kg}$.

This ODE is linear, with standard form

$$P' + \frac{P}{600} = 1$$

We can solve this with variation of parameters. We begin with the homogeneous problem:

$$\begin{aligned} \frac{dP}{dt} + \frac{P}{600} &= 0 \\ \int \frac{600}{P} dP &= -\int dt \\ 600 \ln|P| &= -t + C \\ \implies \quad P_h(t) &= \tilde{C} e^{-t/600} \end{aligned}$$

Promoting $\tilde{C} \to A(t)$ to obtain our ansatz $P(t) = A(t)e^{-t/600}$, we find

$$\begin{aligned} 1 &= P' + \frac{P}{600} \\ &= \left[A(t)e^{-t/600}\right]' + \frac{A(t)}{600} e^{-t/600} \\ &= \left[A'(t)e^{-t/600} - \frac{A(t)}{600} e^{-t/600}\right] + \frac{A(t)}{600} e^{-t/600} \\ &= A'(t)e^{-t/600} \\ \implies \quad A'(t) &= e^{t/600} \\ \implies \quad A(t) &= 600e^{t/600} + C, \end{aligned}$$

so that the general solution to our ODE is

$$P(t) = 600 + Ce^{-t/600}.$$

Setting the initial condition,

$$3000 = P(0) = 600 + C \implies C = 2400.$$

We may now solve for t_*:

$$\begin{aligned} 1200 = P(t_*) &= 600(1 + 4e^{-t_*/600}) \\ \implies \quad 1 &= 4e^{-t_*/600} \\ \implies \quad \boxed{t_* &= 600 \ln(4) \approx 831.78} \end{aligned}$$

□

Example: A recipe recommends baking a chicken breast for 30 minutes after preheating the oven to 400° F. Suppose the oven temperature rises linearly and takes 10 minutes to preheat from when you turn it on at $t = 0$ (measured in minutes). You are impatient and put the chicken in at $t = 0$ instead of waiting for the oven to preheat: how long should you now bake it? Suppose the chicken is cooked upon reaching 165° F, and the room temperature is 75° F.

Newton's law of heating purports that the rate of change of an object's temperature is proportional to its temperature difference with the environment. Denoting the chicken's temperature by $T(t)$,

$$T' = -k(T - Q(t)), \quad T(0) = 75,$$

where $Q(t)$ is the oven temperature, apparently given by

$$Q(t) = \begin{cases} 75 + 32.5t & 0 \le t \le 10 \\ 400 & t \ge 10 \end{cases}$$

This ODE is linear with standard form

$$T' + kT = kQ(t).$$

We first determine k. The recipe's recommendation suggests that the solution to the IVP

$$\tilde{T}' + k\tilde{T} = 400k, \quad T(0) = 75$$

satisfies $\tilde{T}(30) = 165$. This DE is both linear and separable, and the solution is

$$\tilde{T}(t) = 400 - 325e^{-kt}.$$

Hence we have

$$\begin{aligned} 165 = \tilde{T}(30) &= 400 - 325e^{-30k} \\ 235 &= 325e^{-30k} \\ \implies \quad k &= \frac{\ln(325/235)}{30} = \frac{\ln(65/47)}{30} \approx 0.0108. \end{aligned}$$

Returning to the problem for T, we solve the homogeneous version:

$$\begin{aligned} \frac{dT}{dt} + kT &= 0 \\ \implies \quad T_h(t) &= Ce^{-kt} \end{aligned}$$

Promoting $C \to A(t)$ to obtain our ansatz $T(t) = A(t)e^{-kt}$, we find for $0 \le t \le 10$

$$\begin{aligned} kQ(t) &= T' + kT \\ &= \left[A(t)e^{-kt}\right]' + kA(t)e^{-kt} \\ &= A'(t)e^{-kt} \\ \implies \quad A'(t) &= ke^{kt}Q(t) = ke^{kt}(75 + 32.5t) \\ \implies \quad A(t) &= \left(75 + 32.5\left(t - \frac{1}{k}\right)\right)e^{kt} + C, \end{aligned}$$

so that the general solution to the ODE for T on $0 \leq t \leq 10$ is

$$T(t) = 75 + 32.5\left(t - \frac{1}{k}\right) + Ce^{-kt}.$$

Setting the initial condition,

$$75 = T(0) = 75 - \frac{32.5}{k} + C \implies C = \frac{32.5}{k},$$

so we have (for $0 \leq t \leq 10$)

$$T(t) = 75 + 32.5t - \frac{32.5}{k}(1 - e^{-kt}).$$

To compare the time taken to cook, we evaluate

$$T(10) = 400 - \frac{32.5}{k}(1 - e^{-10k}) \approx 91.95$$

and solve for the time t_* at which $\tilde{T}(t) = T(10)$:

$$t_* = \frac{1}{k} \ln\left(\frac{325}{400 - T(10)}\right) \approx 4.955$$

The additional required cook time is then $\boxed{10 - t_* \approx 5.045 \text{ minutes}}$

Compare the temperature solutions for different values of the preheating time in this Desmos demo.

□

Day 11: Existence and uniqueness (1.5.3)

How confident are we that IVP's can be solved? When they can, are solutions unique?

Example: Solve the IVP

$$x' = 3x^{2/3}, \qquad x(t_0) = 0.$$

This nonlinear first order ODE is separable:

$$\begin{aligned} \int x^{-2/3} dx &= \int 3dt \\ 3x^{1/3} &= 3t + C \\ x(t) &= (t + \tilde{C})^3 \end{aligned}$$

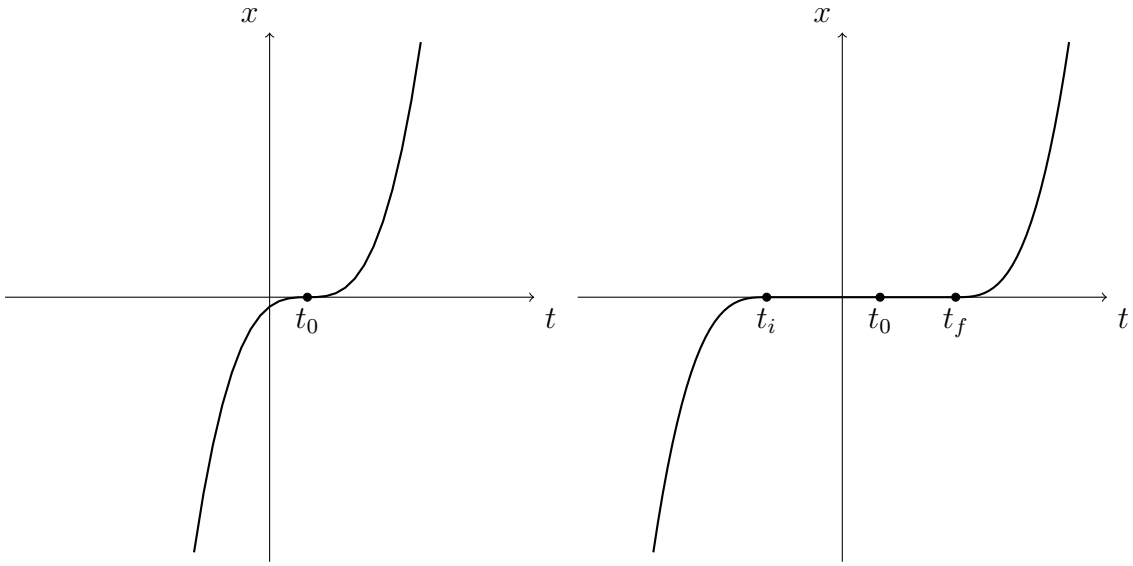
Imposing the initial condition, we find $\tilde{C} = -t_0$, so

$$\boxed{x(t) = (t - t_0)^3}$$

solves the IVP. However, $x(t) = 0$ also satisfies the IVP. In fact, any function of the form

$$x(t) = \begin{cases} (t - t_i)^3 & t \leq t_i \\ 0 & t_i \leq t \leq t_f \\ (t - t_f)^3 & t_f \leq t, \end{cases}$$

for any numbers t_i, t_f satisfying $t_i < t_0 < t_f$, solves the IVP.



Apparently, solutions to this IVP are far from unique.

□

For many physical models, such a situation is unsatisfactory. Fortunately, one can show...

Theorem (linear case): If functions $p(t)$ and $g(t)$ are continuous in the open interval (α, β) containing $t = t_0$, then there exists a unique function $x(t)$ satisfying the IVP

$$x' + p(t)x = g(t), \qquad x(t_0) = x_0$$

on the interval (α, β), for any value x_0.

□

Importantly, one does not have to have found a solution to know that one (and only one) exists, and even to have an idea of where it exists. In the above, α and/or β may be $\pm\infty$.

To find the solution, the method of variation of parameters will always suffice.

Example: Consider the linear ODE

$$(1+t)x' + 2x = 6t.$$

On what intervals must unique solutions exist? Is this born out in the general solution?

In standard form, this ODE reads

$$x' + \frac{2}{1+t}x = \frac{6t}{1+t}.$$

Given the discontinuity at $t = -1$, a unique solution is guaranteed to exist on $(-\infty, -1)$ for any IVP initialized in this range, and similarly for $(-1, \infty)$.

This ODE was in the last example from Day 9, where we found solutions are given by

$$x(t) = \frac{2t^3 + 3t^2 + C}{(1+t)^2}.$$

In general, these are indeed undefined at $t = -1$, as expected.

Note that specifically for $C = -1$, the numerator factors as $2t^3 + 3t^2 - 1 = (1+t)^2(2t-1)$, so that

$$x(t) = 2t - 1$$

is a solution. This specific solution suffers no pathology at $t = -1$.

□

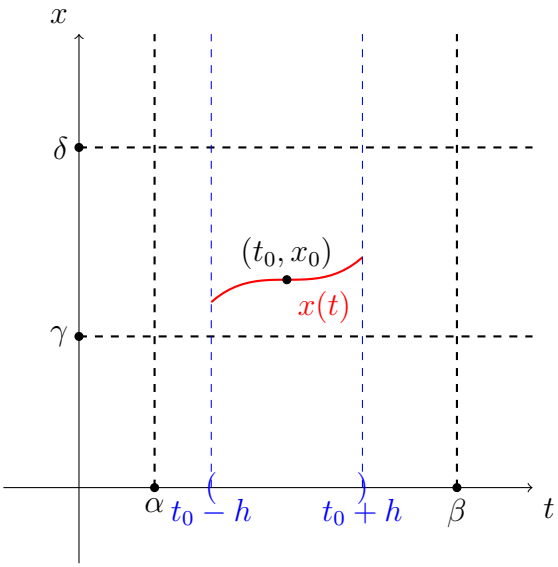
What about the nonlinear case? What went wrong in our first example? One can also show...

Theorem (general case): If a function $f(t, x)$ satisfies that both f and $\frac{\partial f}{\partial x}$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < x < \delta$, containing (t_0, x_0), then there is a unique solution to the IVP

$$x' = f(t, x), \qquad x(t_0) = x_0$$

defined on some interval $(t_0 - h, t_0 + h)$ contained in (α, β).

□



In our first example, $f(t,x) = 3x^{2/3}$ was continuous, but $\frac{\partial f}{\partial x} = 2x^{-1/3}$ had a discontinuity at $x = 0$, resulting in the loss of uniqueness around $x = 0$.

Example: Solve the IVP

$$x' = x^2, \qquad x(0) = x_0.$$

On what interval is this solution defined?

This ODE is separable, and we have

$$\begin{aligned} \int x^{-2} dx &= \int dt \\ -x^{-1} &= t + C \\ x(t) &= \frac{1}{\tilde{C} - t} \end{aligned}$$

Imposing the initial condition, we find $\tilde{C} = \frac{1}{x_0}$, so

$$\boxed{x(t) = \frac{x_0}{1 - x_0 t}}.$$

Note $x(t)$ diverges at $t \to 1/x_0$, so the solution is defined on $(-\infty, \frac{1}{x_0})$ for $x_0 > 0$ and $(\frac{1}{x_0}, \infty)$ for $x_0 < 0$. Note that for large $x_0 > 0$, the blowup time $1/x_0$ can be very small.

□

Important takeaways:

- Existence and uniqueness $\implies$ solutions can't cross (for well-behaved ODE's).
- Expected interval of definition can be read off for linear ODE's; varies widely for nonlinear.
- Linear first-order ODE's admit a general solution with explicit formula (from variation of parameters). Often not true in nonlinear case.

Day 12: Intro to second-order equations (2.1,6.3)

How do we pose and approach 2nd order problems?

We now turn to second-order ODE's in one unknown $x(t)$, having the general form

$$x'' = f(t, x, x'). \tag{7}$$

Example: Show that $x(t) = e^{-t}\sin(t)$ is a solution to

$$x'' = -2(x' + x).$$

We compute

$$\begin{aligned} x(t) &= e^{-t}\sin(t) \\ x'(t) &= -e^{-t}\sin(t) + e^{-t}\cos(t) \\ x''(t) &= -2e^{-t}\cos(t), \end{aligned}$$

so that indeed $-2(x'(t) + x(t)) = -2e^{-t}\cos(t) = x''(t)$.

□

Recall that we could approximate solutions to first-order IVP's via Euler's method.

We divided (t_0, T) into N subintervals between $t_0, t_1, \ldots, t_N$, estimating $x(t_n) \approx X_n$ recursively:

$$X_{n+1} = X_n + x'(t_n)\Delta t,$$

starting from $X_0 = x_0$. While x' was given directly by the DE before, this is no longer true.

Instead, the ODE (7) tells us x'' from x and x'. In each step, then, we will need approximations for both $x(t_n) \approx X_n$ and $x'(t_n) \approx \dot{X}_n$, to be updated in tandem:

$$\boxed{\begin{aligned} X_{n+1} &= X_n + \dot{X}_n\Delta t \\ \dot{X}_{n+1} &= \dot{X}_n + f(t_n, X_n, \dot{X}_n)\Delta t \end{aligned}}$$

A well-posed IVP requires pairing (7) with **two** initial conditions, both $x(t_0) = x_0$ and $x'(t_0) = \dot{x}_0$.

Example: Use Euler's method with $N = 4$ steps to approximate $x(2)$, where $x(t)$ solves the IVP

$$x'' = -2(x' + x), \qquad x(0) = 0, \quad x'(0) = 1.$$

The range from $t_0 = 0$ to $t_4 = 2$ is divided into 4 segments of width $\Delta t = \frac{1}{2}$. We have $X_0 = 0$, $\dot{X}_0 = 1$, and $f(t, x, x') = -2(x' + x)$. We may proceed in a table as before:

n	t_n	X_n	$\dot{X}_n$	$f(t_n, X_n, \dot{X}_n)$	ΔX	$\Delta\dot{X}$
0	0	0	1	-2	0.5	-1
1	0.5	0.5	0	-1	0	-0.5
2	1	0.5	-0.5	0	-0.25	0
3	1.5	0.25	-0.5	$\cdots$	-0.25	$\cdots$
4	2	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$

So we've found the estimate $x(2) \approx X_4 = 0$.

One can see the Euler's method approximation converge to the true solution in this Desmos demo. □

While numerical techniques remain an invaluable tool, qualitative approaches for sketching $x(t)$ are now dramatically less effective. Typically, one can only hope to sketch *local* behavior near t_0.

Example: Sketch the solution $x(t)$ to the IVP

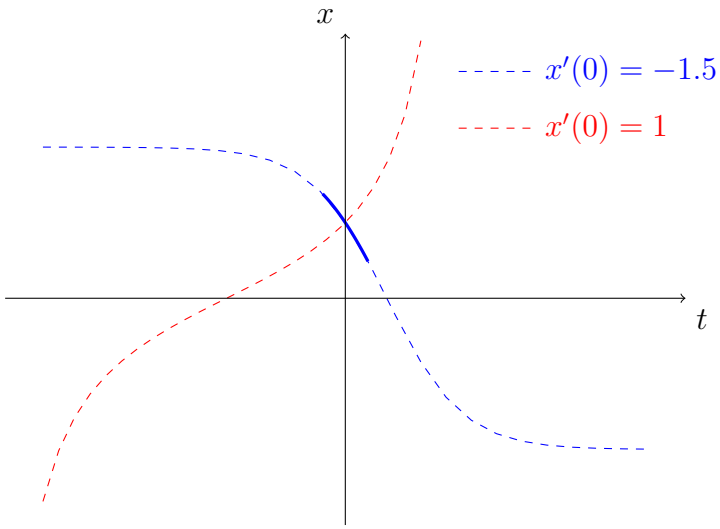
$$x'' = xx', \qquad x(0) = 1,\ x'(0) = -1.5$$

around $t = 0$.

The initial conditions tell us the initial value and slope, and the ODE indicates

$$x''(0) = x(0) \cdot x'(0) = (1) \cdot (-1.5) = -1.5,$$

so the graph is initially concave down:



One could reasonably sketch the solid blue curve, but would be hard-pressed to do the rest. □

Entirely general second-order ODE's (7) are prohibitively difficult to solve. To have hope of constructing solutions, we must specialize.

We will primarily be interested in linear problems: a linear second-order ODE has the standard form

$$x'' + p(t)x' + q(t)x = g(t). \tag{8}$$

This is homogeneous if $g(t) = 0$, and nonhomogeneous otherwise.

The above convergence of Euler's method suggests we can expect existence and uniqueness:

Theorem: Suppose that $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval $I = (\alpha, \beta)$ containing t_0. Then there exists a unique solution $x(t)$ to the IVP

$$x'' + p(t)x' + q(t)x = g(t), \qquad x(t_0) = x_0,\ x'(t_0) = \dot{x}_0$$

on I, for any choice of initial values x_0, $\dot{x}_0$.

□

Day 13: Structure of second-order linear ODE's (2.1,2.2)

How can we construct solutions to second-order equations?

While first-order systems are more general, it is worthwhile to hone in on second-order linear ODE's due to their ubiquity in applications, largely owing to Newton's laws being second-order.

Example: We've seen that the displacement $x(t)$ of a mass m on a spring of stiffness k satisfies

$$mx'' = -kx,$$

and incorporating a linear damping force $F_d = -\gamma x'$ gives

$$mx'' + \gamma x' + kx = 0.$$

This is a second-order linear and homogeneous ODE for the position $x(t)$ of the mass.

It is often of great interest to evaluate this system's response to an external or driving force $F_e(t)$:

$$mx'' + \gamma x' + kx = F_e(t).$$

□

Example: An RLC circuit is a simple electrical circuit containing a resistor with resistance R, inductor with inductance L, capacitor with capacitance C, and a battery supplying voltage $\mathcal{E}$. The laws of E&M dictate that the charge $q(t)$ on the capacitor satisfies the second-order, linear ODE

$$Lq'' + Rq' + \frac{q}{C} = \mathcal{E}.$$

In practice, the supplied voltage often varies with time, $\mathcal{E}(t)$: most power grids supply AC power.

□

Such examples abound, exhibiting the particular utility of second-order linear equations.

As in the first-order case, linear equations enjoy the principle of superposition.

In particular, if $x_1(t), x_2(t)$ both satisfy the homogeneous linear ODE

$$x'' + p(t)x' + q(t)x = 0, \tag{9}$$

then so does any linear combination $x(t) = c_1x_1(t) + c_2x_2(t)$.

Example: Using that $x_1(t) = e^{-t}\sin(t)$ and $x_2(t) = e^{-t}\cos(t)$ each satisfy

$$x'' + 2x' + 2x = 0,$$

solve the IVP consisting of this ODE with the initial conditions $x(0) = 1$, $x'(0) = 1$.

Let us look for a solution of the form $x(t) = c_1x_1(t) + c_2x_2(t)$. We find that

$$x'(t) = c_1x_1' + c_2x_2' = e^{-t}\left[(c_1 - c_2)\cos(t) - (c_1 + c_2)\sin(t))\right],$$

so the initial conditions read

$$\begin{aligned} 1 &= x(0) = c_2 \\ 1 &= x'(0) = c_1 - c_2, \end{aligned}$$

whereby we deduce $c_1 = 2$, $c_1 = 1$, so the solution to our IVP is

$$\boxed{x(t) = e^{-t}\left[2\sin(t) + \cos(t)\right]}$$

□

Crucial takeaway: given two solutions x_1, x_2 to the homogeneous equation (9), the combination

$$x_g := c_1 x_1 + c_2 x_2$$

is dubbed the general solution so long as it can capture all possible initial conditions.

This is ensured if $x_1(t), x_2(t)$ are linearly independent: neither is a constant multiple of the other.

In this event, $x_1(t)$ and $x_2(t)$ are said to comprise a fundamental set.

Recall that an ODE is called autonomous if there is no explicit t dependence: $x'' = f(x, x')$.

When the homogeneous linear equation (9) is autonomous, it reduces to

$$ax'' + bx' + cx = 0. \tag{10}$$

This is sometimes called the case of constant coefficients.

To solve this problem in general, it is apparently enough to identify two independent solutions.

We will do this with an ansatz: we make an educated guess as to the structure of a solution $x(t)$.

For the LHS of (10) to return 0, a cancellation must occur between x, x', and x'', suggesting that these functions should all be similar. A simple ansatz compatible with this expectation is $x(t) = e^{\lambda t}$.

Let us assess whether this ansatz works:

$$\begin{aligned} & 0 = ax'' + bx' + cx \\ & \quad = a(e^{\lambda t})'' + b(e^{\lambda t})' + ce^{\lambda t} \\ & \quad = a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} \\ & \quad = \left[a\lambda^2 + b\lambda + c\right] e^{\lambda t} \\ \iff \quad & 0 = a\lambda^2 + b\lambda + c \end{aligned}$$

That is, our ansatz $x(t) = e^{\lambda t}$ is indeed a solution to (10) so long as λ satisfies

$$a\lambda^2 + b\lambda + c = 0,$$

called the characteristic equation of (10). The LHS is called the characteristic polynomial.

These λ values are the roots of the characteristic polynomial, also known as the ODE's eigenvalues.

To summarize,

Theorem: If λ satisfies the characteristic equation $a\lambda^2 + b\lambda + c = 0$, then $x(t) = e^{\lambda t}$ solves (10),

$$ax'' + bx' + cx = 0.$$

□

The roots of the characteristic polynomial are given by

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

from which we identify the three qualitatively distinct cases:

(1) $b^2 - 4ac > 0$: real and distinct roots λ_1, λ_2.

(2) $b^2 - 4ac = 0$: single real root λ.

(3) $b^2 - 4ac < 0$: complex roots $\lambda_\pm = \mu \pm i\nu$.

We must treat each case.

Day 14: Case of constant coefficients (2.2.1,2.2.2)

We treat the three cases for solving

$$ax'' + bx' + cx = 0,$$

namely

(1) $b^2 - 4ac > 0$: real and distinct roots λ_1, λ_2.

(2) $b^2 - 4ac = 0$: single real root λ.

(3) $b^2 - 4ac < 0$: complex roots $\lambda_\pm = \mu \pm i\nu$.

Case (1): Roots λ_1, λ_2 are real and distinct.

We immediately have a fundamental set of solutions $x_1(t) = e^{\lambda_1 t}$, $x_2(t) = e^{\lambda_2 t}$, so that

$$\boxed{x_g(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

Example: Solve the IVP

$$x'' + x' - 2x = 0, \qquad x(0) = 1, \quad x'(0) = 0.$$

The characteristic equation is

$$0 = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2),$$

giving roots $\lambda_1 = 1$, $\lambda_2 = -2$. These are real and distinct, so the solution takes the form

$$x(t) = c_1 e^t + c_2 e^{-2t}.$$

We then have

$$x'(t) = c_1 e^t - 2c_2 e^{2t},$$

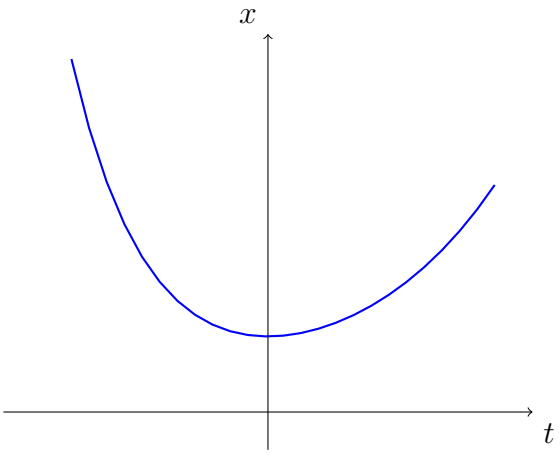
so that the initial conditions impose

$$\begin{aligned} 1 &= x(0) = c_1 + c_2 \\ 0 &= x'(0) = c_1 - 2c_2, \end{aligned}$$

which one may solve to find $c_1 = 2/3$, $c_2 = 1/3$, so that the solution is

$$\boxed{x(t) = \frac{1}{3}\left(2e^t + e^{-2t}\right)}$$

This solution is plotted below.



□

Case (2): Single real, repeated root λ.

The only eigenvalue is $\lambda = -\frac{b}{2a}$, so we know only one solution $x_1(t) = e^{\lambda t}$. We must find another.

If one knows one solution $x_1(t)$ to a homogeneous linear ODE, another may be obtained via reduction of order: we form a new ansatz $x(t) = A(t)x_1(t)$ and substitute,

$$\begin{aligned} 0 &= ax'' + bx' + cx \\ &= a(A(t)x_1)'' + b(A(t)x_1)' + cA(t)x_1 \\ &= ax_1 \cdot A''(t) + (2ax_1' + bx_1) \cdot A'(t) + [ax_1'' + bx_1' + cx_1] \cdot A(t) \\ &= [aA''(t) + (2a\lambda + b)A'(t)]\, e^{\lambda t} \\ &= ae^{\lambda t} A''(t) \\ \implies \quad A''(t) &= 0 \\ \implies \quad A(t) &= c_1 + c_2 t. \end{aligned}$$

Hence our ansatz has told us that the general solution is given by

$$\boxed{x_g(t) = [c_1 + c_2 t]\, e^{\lambda t}}$$

In retrospect, we see that a second independent solution to $x_1(t) = e^{\lambda t}$ was $x_2(t) = te^{\lambda t}$.

Example: Solve the IVP

$$x'' + 4x' + 4x = 0, \qquad x(0) = 1, \quad x'(0) = -1.$$

The characteristic equation is

$$0 = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2,$$

giving the double root $\lambda = -2$. The solution therefore takes the form

$$x(t) = [c_1 + c_2 t]\, e^{-2t}.$$

We then have

$$x'(t) = [c_2 - 2c_1 - 2c_2 t]\, e^{-2t},$$

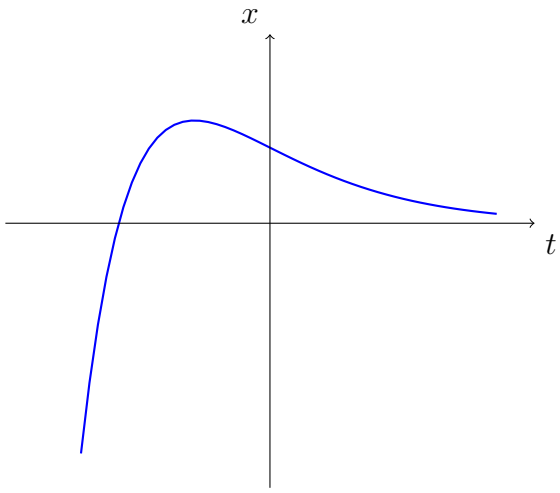
so that the initial conditions impose

$$\begin{aligned} 1 &= x(0) = c_1 \\ -1 &= x'(0) = c_2 - 2c_1, \end{aligned}$$

which yields $c_1 = c_2 = 1$, so that the solution is

$$\boxed{x(t) = (1+t)\, e^{-2t}}$$

This is plotted below.



□

Day 15: Solving linear equations with complex numbers (2.2.3)

Brief Aside on Complex Numbers

We may think of the set of complex numbers $\mathbb{C}$ as the result of appending to the set of real numbers $\mathbb{R}$ a number i with the property that $i^2 = -1$.

We do so not to be whimsical or obtuse, but because it is *useful*. If a, b, and n are positive integers...

Equation	Number type required to solve
$0 = x - a$	Positive Whole
$0 = x + a$	Negative Whole
$0 = ax \pm b$	Rational
$0 = ax^n - b$	Irrational
$0 = ax^n + b$	Complex

It turns out this table stops here– every polynomial equation with rational coefficients has solutions in the complex numbers.

A general complex number is $z = x + iy$, where x and y are real. x is the real part $\mathrm{Re}(z)$, while y is the imaginary part $\mathrm{Im}(z)$. $\bar{z} := x - iy$ is called the complex conjugate to z. These satisfy

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}.$$

Standard arithmetic may be carried out: these may be added, subtracted, multiplied, and divided.

Observe that $z\bar{z}$ is always real and positive:

$$z\bar{z} = (x + iy)(x - iy) = x^2 + ixy - ixy - (iy)^2 = x^2 + y^2.$$

$|z| := \sqrt{z\bar{z}}$ is called the magnitude or absolute value of z.

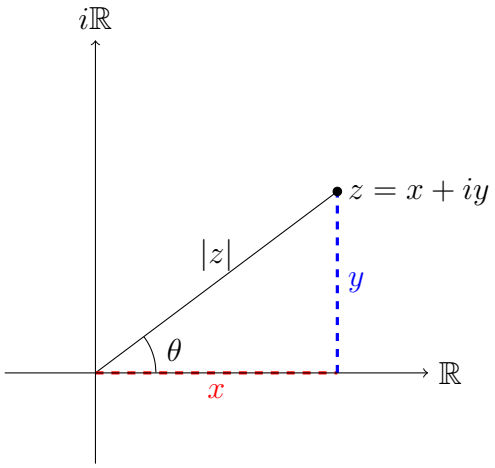
Among the most important identities involving complex numbers is Euler's formula,

$$e^{ix} = \cos(x) + i\sin(x),$$

which can be straightforwardly verified via the Taylor series of e^x, $\cos(x)$, and $\sin(x)$:

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos(x) + i\sin(x). \end{aligned}$$

Any complex number can be written in polar form $z = |z|e^{i\theta}$ for some real number θ.



Case (3): Roots $\lambda_\pm = \mu \pm i\nu$ are complex conjugates.

We immediately have a fundamental set of complex-valued solutions $x_\pm(t) = e^{\lambda_\pm t}$. Expanding,

$$x_\pm(t) = e^{\mu t}e^{\pm i\nu t} = e^{\mu t}\left[\cos(\pm\nu t) + i\sin(\pm\nu t)\right] = e^{\mu t}\left[\cos(\nu t) \pm i\sin(\nu t)\right].$$

Note that $x_\pm(t)$ are always complex conjugates of each other.

To best understand real-valued solutions, we're typically interested in obtaining a fundamental set of real-valued solutions. This can be achieved by taking the real and imaginary parts of $x_+(t)$:

$$x_1(t) := \mathrm{Re}(x_+(t)) = \frac{x_+(t) + x_-(t)}{2}, \qquad x_2(t) := \mathrm{Im}(x_+(t)) = \frac{x_+(t) - x_-(t)}{2i}.$$

More explicitly, these are

$$x_1(t) = e^{\mu t}\cos(\nu t), \qquad x_2(t) = e^{\mu t}\sin(\nu t).$$

Being linear combinations of $x_\pm$, these are solutions (and clearly independent).

Hence a manifestly real presentation of the general solution is

$$\boxed{x_g(t) = e^{\mu t}\left[c_1\cos(\nu t) + c_2\sin(\nu t)\right]}$$

Example: Solve the IVP

$$x'' + 2x' + 26x = 0, \qquad x(0) = 0, \quad x'(0) = 1.$$

The characteristic equation is

$$\lambda^2 + 2\lambda + 26 = 0 \quad \Longrightarrow \quad \lambda_\pm = -1 \pm 5i$$

Hence the solution has the form

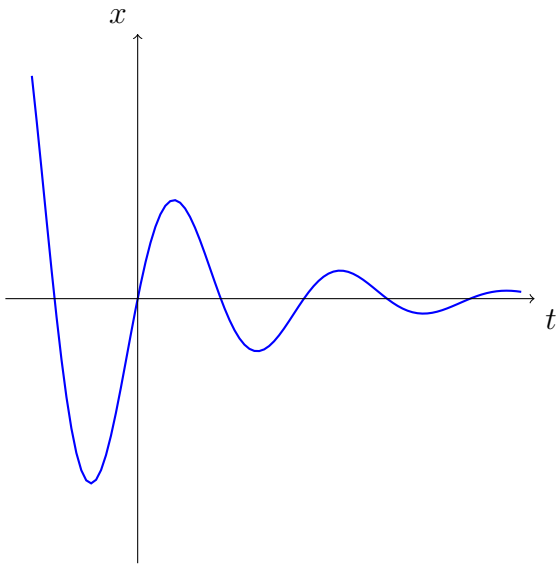
$$\begin{aligned} & x(t) = e^{-t}\left[c_1\cos(5t) + c_2\sin(5t)\right] \\ \Longrightarrow\ & x'(t) = e^{-t}\left[(5c_2 - c_1)\cos(5t) - (5c_1 + c_2)\sin(5t)\right] \end{aligned}$$

The initial conditions impose

$$0 = x(0) = c_1,$$
$$1 = x'(0) = 5c_2 - c_1,$$

giving $c_1 = 0$, $c_2 = 1/5$, so that the solution is

$$\boxed{x(t) = \frac{e^{-t}}{5}\sin(5t).}$$



□

Day 16: Oscillators (2.2.4)

What do solutions to second order ODE's tell us about mechanical oscillators?

Recall that a damped, driven simple harmonic oscillator, e.g. mass on a spring, is governed by

$$mx'' + \gamma x' + kx = F(t),$$

where $m, \gamma, k > 0$ are the mass, damping coefficient, and spring constant; $F(t)$ is the driving force.

For today, we consider undriven oscillators, $F(t) = 0$. We begin with the undamped case, $\gamma = 0$:

$$mx'' + kx = 0.$$

Dividing through by m and setting $\omega_0^2 := k/m$, this is

$$x'' + \omega_0^2 x = 0.$$

The characteristic equation of this ODE is

$$\lambda^2 + \omega_0^2 = 0,$$

with pure imaginary roots $\lambda_\pm = \pm i\omega_0$. The general solution is therefore

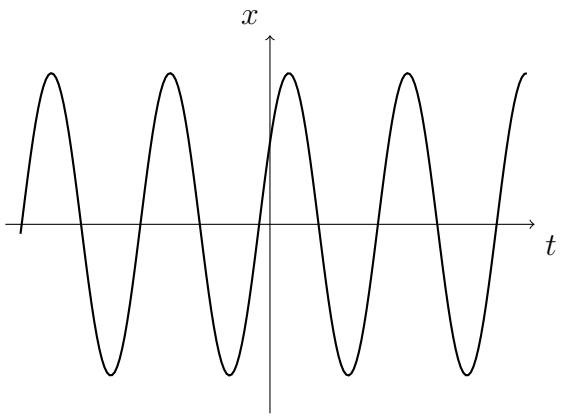
$$x_g(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

It is worthwhile to cast this in the equivalent phase-amplitude form

$$x_g(t) = C\cos(\omega_0 t - \delta),$$

where $C = \sqrt{A^2 + B^2}$ is the amplitude, and the phase δ satisfies $\cos(\delta) = \frac{A}{C}$, $\sin(\delta) = \frac{B}{C}$.

This is a shifted cosine wave with amplitude C and frequency ω_0, indicating how the mass oscillates:



$\omega_0 := \sqrt{k/m}$ is called the natural frequency of the oscillator, and the associated period T_0 is

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}.$$

We now include damping, so we consider

$$mx'' + \gamma x' + kx = 0.$$

Note that the unique critical point is $x = 0$, the oscillator's equilibrium position.

This is a 2nd-order, linear, homogeneous, autonomous ODE with characteristic equation

$$m\lambda^2 + \gamma\lambda + k = 0 \qquad \Longrightarrow \qquad \lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

As in our general treatment of such equations, there are three scenarios.

(1) **Overdamped Harmonic Motion**: $(\gamma^2 - 4mk > 0)$:
There are two distinct, negative, real roots λ_1, λ_2, so the general solution is

$$x_g(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}.$$

(2) **Critically Damped Harmonic Motion** $(\gamma^2 - 4mk = 0)$:
There is a single root $\lambda = -\gamma/2m$, so the general solution is

$$x_g(t) = e^{-\gamma t/2m}\left[A + Bt\right].$$

(3) **Underdamped Harmonic Motion** $(\gamma^2 - 4mk < 0)$:
The roots are complex, $\lambda_\pm = \mu \pm i\nu$, where

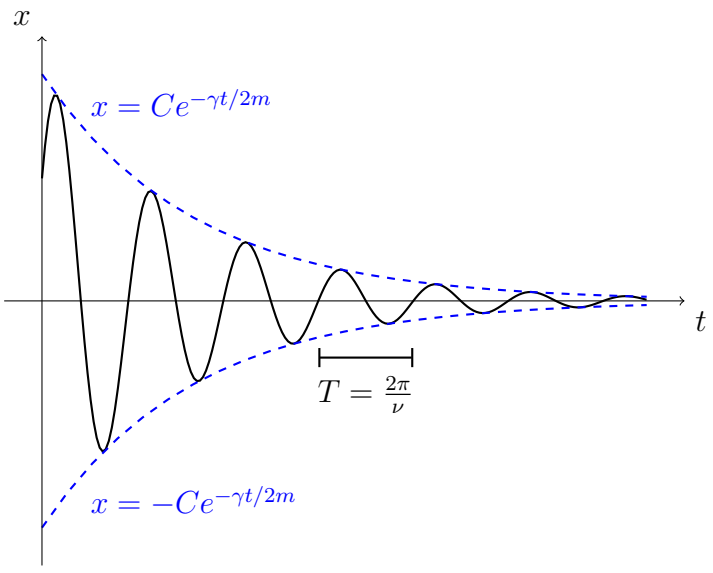
$$\mu = -\frac{\gamma}{2m}, \qquad \nu = \frac{\sqrt{4mk - \gamma^2}}{2m} = \omega_0\sqrt{1 - \frac{\gamma^2}{4km}},$$

so the general solution is

$$\begin{aligned} x_g(t) &= e^{-\gamma t/2m}\left[A\cos(\nu t) + B\sin(\nu t)\right] \\ &= Ce^{-\gamma t/2m}\cos(\nu t - \delta). \end{aligned}$$

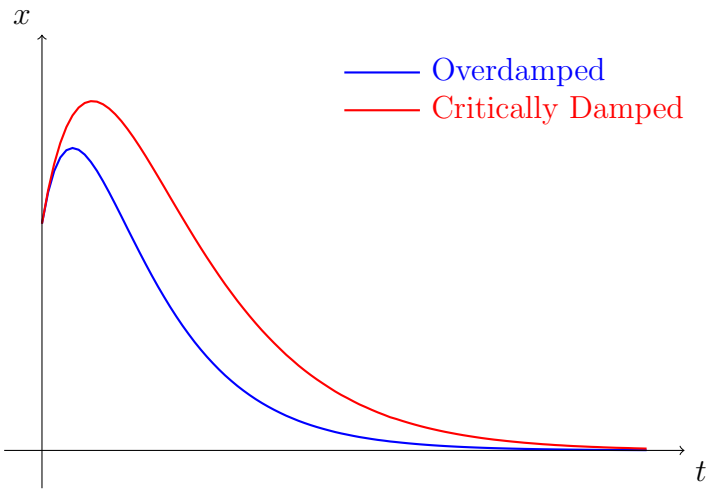
Solutions are subject to decay in all cases, as damping extracts energy from the oscillator.

In the underdamped case (3), solutions oscillate with quasi-frequency ν while they decay:



This corresponds to an oscillation that dies out, e.g. a playground swing reaching lower and lower.

The overdamped case (1) yields exponential decay– damping is strong enough that the system returns to equilibrium without oscillating. The critically damped case (2) is qualitatively similar:



Above are overdamped and critically damped solutions with the same initial condition and limiting exponential decay rate. The overdamped solution has a lower maximum and tends to zero faster.

These yield a slowed, direct return to equilibrium, e.g. a spring door with a damping mechanism to avoid slamming shut, or a car's suspension absorbing shocks/bumps.

Example: A mass of 1 kg is attached to a vertical spring, causing the spring to stretch 20 cm to its new equilibrium. The spring is then stretched by 14 cm from its new equilibrium and released. If friction and drag impart 2 N of force per m/s of speed, find the spring's displacement from equilibrium $x(t)$ in cm, as a function of time. How long does it take for the oscillation amplitude to die down to 1 cm? You may approximate $g = 10$ m/s^2.

We have $m = 1$ kg and $\gamma = 2\,\frac{\text{N}}{\text{m/s}} = 2\,\frac{\text{kg}}{\text{s}}$, and Hooke's law indicates

$$k \cdot (20 \text{ cm}) = mg = 10 \text{ N},$$

so that $k = 0.5\,\frac{\text{N}}{\text{cm}} = 50\,\frac{\text{kg}}{\text{s}^2}$. This gives the IVP

$$x'' + 2x' + 50x = 0, \qquad x(0) = 14, \quad x'(0) = 0$$

and the characteristic equation has roots

$$\lambda_\pm = \frac{-2 \pm \sqrt{4 - 200}}{2} = -1 \pm 7i$$

This is underdamped, and we have

$$\begin{aligned} x(t) &= e^{-t}\left[A\cos(7t) + B\sin(7t)\right] \\ x'(t) &= e^{-t}\left[(7B - A)\cos(7t) - (7A + B)\sin(7t)\right] \end{aligned}$$

Setting the initial conditions, we find

$$\begin{aligned} 14 &= A \\ 0 &= 7B - A, \end{aligned}$$

so that $A = 14$, $B = 2$. We have therefore found

$$\boxed{x(t) = e^{-t}\left[14\cos(7t) + 2\sin(7t)\right]}$$

The effective oscillation amplitude is Ce^{-t} with $C = \sqrt{A^2 + B^2} = \sqrt{200}$, which decays to 1 at time $t_* = \ln(200)/2 \approx \boxed{2.65 \text{ seconds}}$.

Hit play on the t slider in this Desmos demo to visualize this solution.

□

Day 17: Method of undetermined coefficients (2.3.1)

How can we construct solutions to nonhomogeneous equations?

We've said all we can about solutions to linear, homogeneous, and autonomous 2nd-order equations.

What more can be said about a general linear second-order equation (8)?

$$x'' + p(t)x' + q(t)x = g(t) \qquad \text{((8) revisited)}$$

Recall that if x_a and x_b are solutions to a linear DE with source $g(t)$, then the principle of superposition tells us that $x_a - x_b$ is a solution to the corresponding homogeneous problem.

This has a far-reaching implication:

Corollary: If x_1, x_2 comprise a fundamental set of solutions to the homogeneous problem and $x_p(t)$ is some particular solution to (8), then every solution to (8) has the form

$$\begin{aligned} x_g &= c_1 x_1 + c_2 x_2 + x_p \\ &= x_h + x_p \end{aligned}$$

□

$x_g := x_h + x_p$ is therefore the general solution to the nonhomogeneous equation (8).

Solving (8) in general, then, amounts to solving the homogeneous problem in general and finding *some* single solution x_p to (8).

We can construct x_h, for example, in the case of constant coefficients. How might we find an x_p?

One strategy, the method of undetermined coefficients, essentially amounts to informed guessing.

Example: Find a particular solution x_p to

$$x'' + 3x' - 4x = e^{2t},$$

and construct the general solution $x_g(t)$.

Since differentiating e^{2t} yields something similar to itself, a plausible guess is $x_p(t) = Ae^{2t}$. Check:

$$\begin{aligned} e^{2t} &\stackrel{?}{=} x_p'' + 3x_p - 4x_p \\ &= 4Ae^{2t} + 6Ae^{2t} - 4e^{2t} \\ &= 6Ae^{2t} \end{aligned}$$

To satisfy this equality, we must choose $A = 1/6$, giving that $\boxed{x_p(t) = \frac{e^{2t}}{6}}$ is a working choice.

We now simply append x_p to the general solution $x_h(t) = c_1 e^t + c_2 e^{-4t}$ to the homogeneous equation,

$$x'' + 3x' - 4x = 0,$$

to find the general solution to the original problem:

$$\boxed{x_g(t) = c_1 e^t + c_2 e^{-4t} + \frac{1}{6} e^{2t}}$$

□

In general, we formulate a guess for the structure of x_p that might plausibly return $g(t)$ in the ODE.

This is typically only reasonably doable if the homogeneous problem has constant coefficients.

Example: Find a particular solution x_p to

$$x'' - 5x' + 6x = \sin(3t).$$

Differentiating $\sin(3t)$ can return itself, but it also yields $\cos(3t)$ terms. To ensure we can cancel these and get the $\sin(3t)$ term right, we need two constants: $x_p = A\sin(3t) + B\cos(3t)$:

$$\begin{aligned}\sin(3t) &\stackrel{?}{=} x_p'' - 5x_p' + 6x_p \\ &= -9A\sin(3t) - 9B\cos(3t) - 15A\cos(3t) + 15B\sin(3t) + 6A\sin(3t) + 6B\cos(3t) \\ &= [15B - 3A]\sin(3t) - [15A + 3B]\cos(3t)\end{aligned}$$

To satisfy this equality, we need

$$\begin{aligned}15B - 3A &= 1 \\ 15A + 3B &= 0\end{aligned} \implies A = -1/78,\ B = 5/78,$$

giving

$$\boxed{x_p(t) = \frac{5\cos(3t) - \sin(3t)}{78}}$$

□

Example: Find a particular solution x_p to

$$x'' + 2x' + x = t^2 - t$$

Putting $t^2 - t$ into the ODE yields t^2, t, and constant terms. Each should be included in our guess, $x_p = At^2 + Bt + C$:

$$\begin{aligned}t^2 - t &\stackrel{?}{=} x_p'' + 2x_p' + x_p \\ &= 2A + 2(2At + B) + At^2 + Bt + C \\ &= At^2 + (4A + B)t + (2A + 2B + C).\end{aligned}$$

This requires

$$\begin{aligned}A &= 1 \\ 4A + B &= -1 \\ 2A + 2B + C = 0 \quad &\implies \quad A = 1,\ B = -5,\ C = 8,\end{aligned}$$

giving

$$\boxed{x_p(t) = t^2 - 5t + 8}$$

□

If $g(t)$ is a product of these types, e.g. $p(t)e^{\alpha t}\cos(\beta t)$, one may take products of the above guesses:

Example: If $g(t) = t^3 e^t \sin(2t)$, one might take the ansatz

$$x_p(t) = (A_3t^3 + A_2t^2 + A_1t + A_0)e^t\sin(2t) + (B_3t^3 + B_2t^2 + B_1t + B_0)e^t\cos(2t).$$

□

Essentially, **every type of term that shows up upon differentiation should already be included in the ansatz**.

Day 18: Method of undetermined coefficients (cont'd)

An obstruction may arise if the natural guess is a solution to the homogeneous equation:

Example: Find a particular solution x_p to

$$x'' + 2x' + x = 2e^{-t}$$

Following previous examples, we might take $x_p(t) = Ae^{-t}$:

$$\begin{aligned} 2e^{-t} &\stackrel{?}{=} x_p'' + 2x_p' + x_p \\ &= Ae^{-t} - 2Ae^{-t} + Ae^{-t} \\ &= 0. \end{aligned}$$

This is never satisfied, regardless of the choice of A. However, if we choose $x_p(t) = At^2e^{-t}$, we find

$$\begin{aligned} 2e^{-t} &\stackrel{?}{=} x_p'' + 2x_p' + x_p \\ &= \left(At^2e^{-t} - 4Ate^{-t} + 2Ae^{-t}\right) + \left(4Ate^{-t} - 2At^2e^{-t}\right) + At^2e^{-t} \\ &= 2Ae^{-t}, \end{aligned}$$

which can be satisfied by choosing $A = 1$, giving

$$\boxed{x_p(t) = t^2e^{-t}}$$

□

If one's guess solves the homogeneous problem, multiply by the lowest power of t such that this is no longer the case. Above, te^{-t} still solved the homogeneous problem, so we proceeded to t^2e^{-t}.

The superposition principle ensures that we can find a solution x_p for a source $g_1 + g_2$ by finding solutions x_{p1} and x_{p2} for g_1 and g_2 separately.

Example: Find the general solution to

$$x'' - x' - 6x = 36t + 10e^{3t}.$$

First observe that the general solution to the homogeneous problem is

$$x_h(t) = c_1e^{3t} + c_2e^{-2t}.$$

We construct particular solutions x_{p1} and x_{p2} corresponding to $g_1(t) = 36t$ and $g_2(t) = 10e^{3t}$.

We take

$$x_{p1}(t) = At + B$$

and, noting that e^{3t} solves the homogeneous problem,

$$x_{p2}(t) = Cte^{3t}.$$

Checking that substituting x_{p1} yields g_1, we obtain

$$\begin{aligned} 36t = g_1(t) &\stackrel{?}{=} x_{p1}'' - x_{p1}' - 6x_{p1} \\ &= -6At - A - 6B \end{aligned}$$

This yields the system of equations

$$\begin{aligned} 36 &= -6A \\ 0 &= A + 6B \end{aligned} \qquad \Longrightarrow \qquad A = -6,\ B = 1$$

Checking that substituting x_2 yields g_2, we obtain,

$$\begin{aligned} 10e^{3t} = g_2(t) &\stackrel{?}{=} x_{p2}'' - x_{p2}' - 6x_{p2} \\ &= 5Ce^{3t}, \end{aligned}$$

giving $C = 2$. Putting this together, we've found

$$\begin{aligned} x_g(t) &= x_h(t) + x_{p1}(t) + x_{p2}(t) \\ &= c_1e^{3t} + c_2e^{-2t} - 6t + 1 + 2te^{3t} \\ &= \boxed{(c_1 + 2t)e^{3t} + c_2e^{-2t} - 6t + 1} \end{aligned}$$

□

Day 19: Forced oscillators (2.3.2)

With the tool of undetermined coefficients, we return to the damped, driven harmonic oscillator:

$$mx'' + \gamma x' + kx = F(t), \tag{11}$$

For later simplicity, we'll divide through by m and introduce $\omega_0^2 := k/m$ and $b := \gamma/(2m)$,

$$x'' + 2bx' + \omega_0^2 x = f(t).$$

We can now assess how the oscillator responds to sinusoidal driving, $f(t) = A\cos(\omega t) + B\sin(\omega t)$.

This problem is treated most elegantly by taking $B = iA$, giving $f(t) = Ae^{i\omega t}$.

With this choice of f, we seek a particular solution through the ansatz $x_p = Ce^{i\omega t}$ as usual:

$$\begin{aligned} Ae^{i\omega t} &\stackrel{?}{=} x_p'' + 2bx_p' + \omega_0^2 x_p \\ &= -C\omega^2 e^{i\omega t} + 2iC\omega b e^{i\omega t} + C\omega_0^2 e^{i\omega t} \\ &= \left[\omega_0^2 - \omega^2 + 2i\omega b\right] Ce^{i\omega t}, \end{aligned}$$

by which we conclude that the appropriate choice of C is

$$C = \frac{A}{\omega_0^2 - \omega^2 + 2i\omega b},$$

so we have found

$$x_p(t) = \frac{Ae^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\omega b} \tag{12}$$

As usual, the general solution to (11) is now given by

$$x_g(t) = x_h(t) + x_p(t).$$

Recall from Day 16 that x_h for a damped oscillator is always subject to exponential decay, so that $\lim_{t\to\infty} x_h(t) = 0$. The homogeneous piece x_h of x_g is therefore called the transient solution.

By the same token, $x_g(t) \approx x_p(t)$ after a sufficiently long time, regardless of initial conditions. x_p is called the steady-state solution (or the system's response or output).

In many applications, transient solutions die out quickly, making x_p crucial to understand.

Example: Solve the IVP,

$$x'' + 2x' + 50x = 145\cos(5t), \qquad x(0) = 15,\ x'(0) = 0. \tag{13}$$

From the example on Day 16, we know that the general solution x_h to the homogeneous problem is

$$x_h(t) = e^{-t}\left[c_1\cos(7t) + c_2\sin(7t)\right]$$

Noting that $145\cos(5t) = \text{Re}(145e^{i5t})$, we take the real part of (12) to obtain x_p here:

$$\begin{aligned} x_p(t) &= \text{Re}\left(\frac{145e^{i5t}}{\omega_0^2 - \omega^2 + 2i\omega b}\right) \\ &= \text{Re}\left(\frac{145e^{i5t}}{25 + 10i}\right) = \text{Re}\left(\frac{145e^{i5t}}{25 + 10i} \cdot \frac{25 - 10i}{25 - 10i}\right) \\ &= \text{Re}\left(\frac{145e^{i5t}(25 - 10i)}{25^2 + 10^2}\right) = \frac{145}{725}\left(25\cos(5t) + 10\sin(5t)\right) \\ &= 5\cos(5t) + 2\sin(5t). \end{aligned}$$

The general solution is $x_g = x_h + x_p$, so $x(t)$ has the form

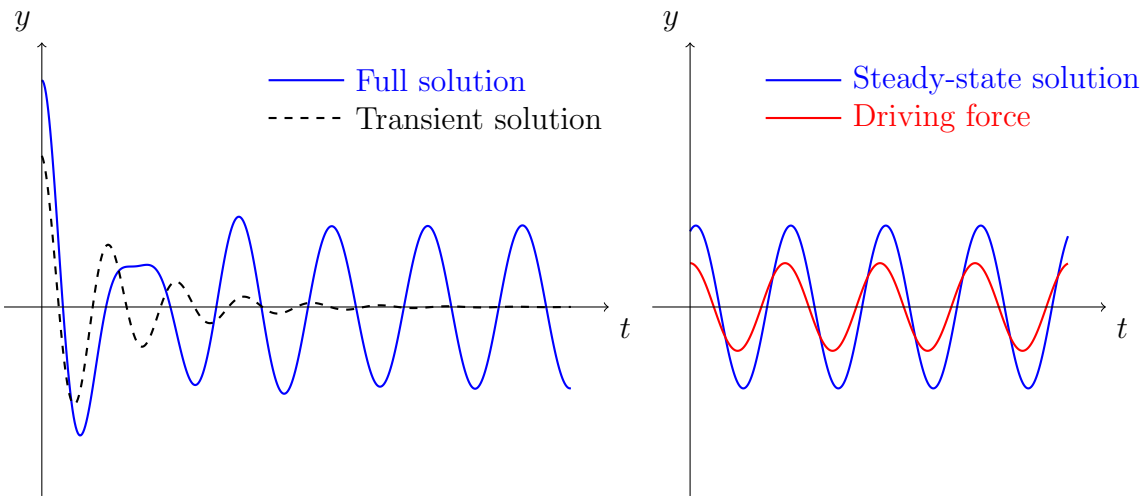
$$\begin{aligned} x(t) &= e^{-t}\left[c_1\cos(7t) + c_2\sin(7t)\right] + 5\cos(5t) + 2\sin(5t) \\ \implies x'(t) &= e^{-t}\left[(7c_2 - c_1)\cos(7t) - (7c_1 + c_2)\sin(7t)\right] - 25\sin(5t) + 10\cos(5t), \end{aligned}$$

and setting the initial conditions now yields

$$\begin{aligned} 15 &= x(0) = c_1 + 5 \\ 0 &= x'(0) = 7c_2 - c_1 + 10 \end{aligned} \qquad \implies \qquad c_1 = 10,\ c_2 = 0,$$

so that the solution to the IVP is

$$x(t) = 10e^{-t}\cos(7t) + 5\cos(5t) + 2\sin(5t)$$



While the driving force here is a pure cosine wave, the steady-state response is a combination of sine and cosine. That is, there's a phase offset between the input and output.

Hit play on the t slider in this Desmos demo to visualize this response.

□

Day 20: Resonance (2.3.2)

Continuing our discussion of forced oscillators, it's useful to quantify the steady-state (12)'s relation to the driving force in the frequency response

$$G(\omega) := \frac{x_p(t)}{Ae^{i\omega t}} = \frac{1}{\omega_0^2 - \omega^2 + 2i\omega b},$$

a function of the driving frequency ω. Being complex, the response may be written in polar form,

$$G(\omega) = |G(\omega)|e^{-i\phi(\omega)},$$

where the gain $|G(\omega)|$, the ratio between the steady-state and driving amplitudes, is given by

$$|G(\omega)| = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 b^2}},$$

and the phase $\phi(\omega)$, the phase lag between the steady-state and the driving signals, is given by

$$\phi(\omega) = \cos^{-1}\left(\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 b^2}}\right).$$

These interpretations are justified by the definitional relation

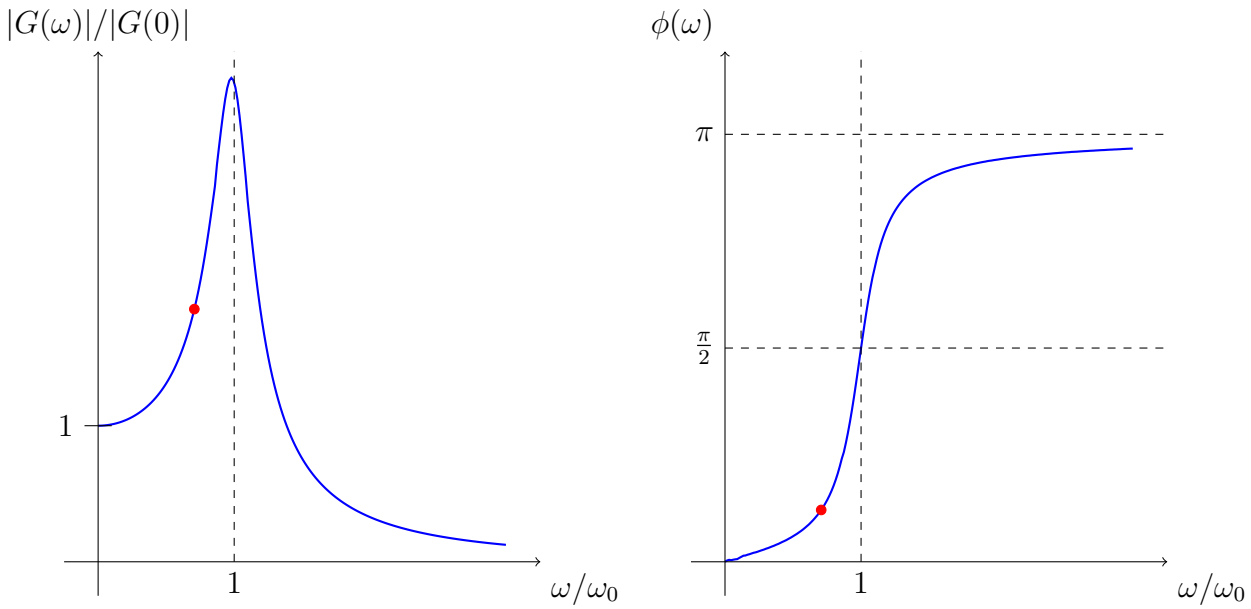
$$x_p(t) = G(\omega) \cdot Ae^{i\omega t} = A|G(\omega)|e^{i(\omega t - \phi(\omega))}.$$

Indeed, taking real and imaginary parts with $A = 1$ leads to the mapping

Input		Output
$\cos(\omega t)$	$\mapsto$	$\lvert G(\omega)\rvert \cos(\omega t - \phi(\omega))$
$\sin(\omega t)$		$\lvert G(\omega)\rvert \sin(\omega t - \phi(\omega))$

between a driving force input and the steady-state output.

The gain and phase for the example system (13), wherein $b = 1$ and $\omega_0^2 = 50$, are:



The red points sit at $\omega = 5$, as in the example's driving force.

The gain function above has a clear, sharp maximum near $\omega \approx \omega_0$: this exhibits the significant physical phenomenon of resonance, with the maximum occurring at the resonant frequency.

An oscillator system can exhibit resonance if the damping is small, in the sense $\frac{b}{\omega_0} = \frac{\gamma}{\sqrt{km}} << 1$.

To visualize the phenomenon of resonance in the example system, hit play on the t slider and vary the driving frequency ω in this Desmos demo. Note that $\omega_0 = \sqrt{50} \approx 7.07$.

In the absence of damping and taking a cosine driving force, (11) becomes

$$x'' + \omega_0^2 x = A\cos(\omega t).$$

Much of the same analysis goes through by simply setting $b = 0$: the general solution is

$$x_g(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t).$$

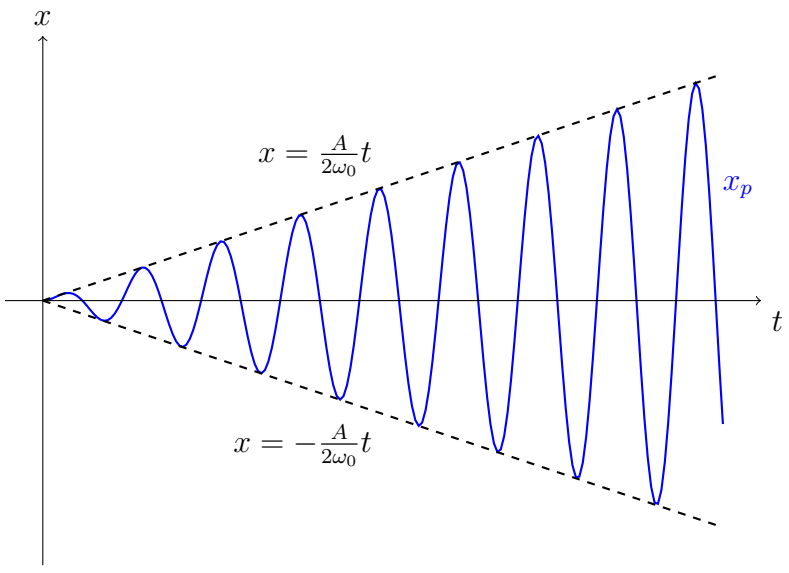
The homogeneous solution isn't transient, but the response still exhibits resonance near $\omega \approx \omega_0$.

We must update our approach for $\omega = \omega_0$, as the above x_p now solves the homogeneous equation.

Carrying through the method of undetermined coefficients then yields

$$x_g(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{A}{2\omega_0}t\sin(\omega_0 t).$$

Notice that the amplitude of x_p now grows without bound:



Without sufficient damping to curtail it, resonance eventually leads to a breakdown of the system.

Day 21: Variation of Parameters (2.4.2); Reduction of Order (2.4.3)

How can we construct solutions to nonhomogeneous equations more generally?

Undetermined coefficients is useful, but limited; it is complemented by variation of parameters.

Given a general second-order linear ODE

$$x'' + p(t)x' + q(t)x = g(t), \qquad \text{((8) revisited)}$$

suppose that we already know a fundamental set of solutions x_1, x_2 to the homogeneous problem, so that $x_h = c_1x_1 + c_2x_2$, and we would like to construct the general solution x_g.

Idea of variation of parameters: promote the constants c_1, c_2 in x_h to functions, giving the ansatz

$$x_g(t) = A(t)x_1(t) + B(t)x_2(t),$$

with $A(t)$, $B(t)$ unknown functions to determine. That our ansatz must satisfy (8) imposes one constraint on these two unknowns. To solve, we must impose another. Noting

$$x_g'(t) = A(t)x_1' + B(t)x_2' + A'(t)x_1 + B'(t)x_2,$$

we see that a simplifying constraint to impose might be requiring

$$A'(t)x_1 + B'(t)x_2 = 0.$$

In this event, x_g' and x_g'' become

$$\begin{aligned} x_g' &= A(t)x_1' + B(t)x_2' \\ x_g'' &= A(t)x_1'' + A'(t)x_1' + B(t)x_2'' + B'(t)x_2'. \end{aligned}$$

Plugging these into (8) then yields

$$\begin{aligned} g(t) &\stackrel{?}{=} x_g'' + g(t)x_g' + q(t)x_g \\ &= A'(t)x_1' + A(t)\left[x_1'' + p(t)x_1' + q(t)x_1\right] \\ &\quad + B'(t)x_2' + B(t)\left[x_2'' + p(t)x_2' + q(t)x_2\right] \\ &= A'(t)x_1' + B'(t)x_2' \end{aligned}$$

That is, a pair of conditions on $A(t)$, $B(t)$ that ensures our ansatz solves (8) is apparently

$$\boxed{\begin{aligned} 0 &= A'(t)x_1 + B'(t)x_2 \\ g(t) &= A'(t)x_1' + B'(t)x_2' \end{aligned}}$$

These can always be solved algebraically for $A'(t)$ and $B'(t)$, which can then be integrated.

Example: Find the general solution to

$$x'' + 4x' + 3x = \frac{e^{-t}}{1 + e^{2t}}.$$

The characteristic equation for the homogeneous problem is

$$0 = \lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1),$$

so a fundamental set is

$$x_1(t) = e^{-3t}, \qquad x_2(t) = e^{-t},$$

and our ansatz takes the form

$$x_g(t) = A(t)x_1(t) + B(t)x_2(t).$$

Imposing the conditions boxed above,

$$\begin{aligned} A'x_1 + B'x_2 = 0 \quad &\implies \quad e^{-3t}A' + e^{-t}B' = 0 \\ A'x_1' + B'x_2' = g(t) \implies &-3e^{-3t}A' - e^{-t}B' = g(t) \end{aligned}$$

We solve these algebraically for A', B' and integrate. Adding the equations isolates A', giving

$$A'(t) = -\frac{e^{3t}g(t)}{2} \implies A(t) = -\frac{1}{2}\int \frac{e^{2t}}{1+e^{2t}}dt = -\frac{1}{4}\ln\left(1+e^{2t}\right) + c_1.$$

Similarly isolating B', we find

$$B'(t) = \frac{e^{t}g(t)}{2} \implies B(t) = \frac{1}{2}\int \frac{1}{1+e^{2t}}dt = -\frac{1}{4}\ln\left(1+e^{-2t}\right) + c_2.$$

Our ansatz therefore yields

$$\boxed{x_g(t) = c_1e^{-3t} + c_2e^{-t} - \frac{e^{-3t}}{4}\ln\left(1+e^{2t}\right) - \frac{e^{-t}}{4}\ln\left(1+e^{-2t}\right)}$$

□

How can we find homogeneous solutions with non-constant coefficients?

There is no universal method to construct the solution to a general homogeneous problem,

$$x'' + p(t)x' + q(t)x = 0. \tag{14}$$

If one can *somehow* identify one solution $x_1(t)$, though, all others follow from reduction of order:

We take the ansatz $x_h(t) = A(t)x_1(t)$ and substitute, requiring

$$\begin{aligned} 0 &= x_h'' + p(t)x_h' + q(t)x_h \\ &= A''(t)x_1 + 2A'(t)x_1' + p(t)A'(t)x_1 + A(t)\left[x_1'' + p(t)x_1' + q(t)x_1\right] \\ &= A''(t)x_1 + 2A'(t)x_1' + p(t)A'(t)x_1. \end{aligned}$$

Setting $y(t) := A'(t)$, we see that it must satisfy the **first-order** linear, homogeneous equation

$$x_1(t)y' + (2x_1'(t) + p(t)x_1(t))y = 0,$$

which can always be solved via separation of variables.

Example: Find the general solution to

$$x'' + (\tan(t) - 1)x' - \tan(t)x = 0,$$

using that one solution is given by $x_1(t) = e^t$.

We consider the ansatz $x_h(t) = A(t)x_1(t) = A(t)e^t$ and plug in:

$$\begin{aligned} & 0 = x_h'' + (\tan(t) - 1)x_h' - \tan(t)x_h \\ & \quad = \left[A'' + 2A' + (\tan(t) - 1)A'\right] e^t \\ \iff \qquad & 0 = A'' + (1 + \tan(t))A'. \end{aligned}$$

We can now solve for $y(t) := A'(t)$ via separation of variables,

$$\begin{aligned} & y' = -(1 + \tan(t))y \\ \implies \qquad & \int \frac{dy}{y} = -\int (1 + \tan(t))dt \\ & \ln|y| = \ln|\cos(t)| - t + C \\ \implies \qquad & y(t) = \tilde{C}\cos(t)e^{-t} \\ \implies \qquad & A(t) = \int y(t)dt = c_1 + c_2\left(\sin(t) - \cos(t)\right)e^{-t}. \end{aligned}$$

Plugging this into our ansatz, the general solution to this homogeneous problem is

$$\boxed{x_h(t) = c_1 e^t + c_2(\sin(t) - \cos(t))}$$

□

Example: Using that $x_1(t) = \frac{\cos(t)}{t}$ solves

$$t^2x'' + 2tx' + t^2x = 0,$$

solve the IVP

$$t^2x'' + 2tx' + t^2x = t, \qquad x(\pi) = 0,\ x'(\pi) = 0.$$

We first find the full homogeneous solution $x_h(t)$ by taking the ansatz $x_h(t) = \tilde{A}(t)x_1(t) = \tilde{A}(t)\frac{\cos(t)}{t}$:

$$\begin{aligned} 0 &= t^2x_h'' + 2tx_h' - t^2x_h \\ &= t^2\left(\frac{\cos(t)}{t}\right)\tilde{A}'' + 2t^2\left(\frac{\cos(t)}{t}\right)'\tilde{A}' + 2t\left(\frac{\cos(t)}{t}\right)\tilde{A}' \\ &= t\cos(t)\tilde{A}'' - 2t\sin(t)\tilde{A}', \end{aligned}$$

and we solve for $y(t) := \tilde{A}'$ by separation of variables:

$$\begin{aligned} & y' = 2\tan(t)y \\ \implies \quad & \int \frac{dy}{y} = 2\int \tan(t)dt \\ & \ln|y| = -2\ln|\cos(t)| + C \\ \implies \quad & y(t) = \tilde{C}\sec^2(t) \\ \implies \quad & \tilde{A}(t) = \int y(t)dt = c_1 + c_2\tan(t). \end{aligned}$$

Hence the homogeneous solution is

$$x_h(t) = \tilde{A}(t)\frac{\cos(t)}{t} = c_1\frac{\cos(t)}{t} + c_2\frac{\sin(t)}{t}.$$

In particular, a fundamental set of homogeneous solutions is $x_1(t) = \frac{\cos(t)}{t}$, $x_2(t) = \frac{\sin(t)}{t}$.

To solve the full problem, we therefore form the ansatz

$$x_g(t) = A(t)\frac{\cos(t)}{t} + B(t)\frac{\sin(t)}{t},$$

where $A(t)$ and $B(t)$ are determined by the constraints

$$\begin{aligned} A'x_1 + B'x_2 = 0 \quad &\implies \quad \cos(t)A' + \sin(t)B' = 0 \\ A'x_1' + B'x_2' = g(t) \quad &\implies \quad -(t\sin(t) + \cos(t))A' + (t\cos(t) - \sin(t))B' = t^2 g(t), \end{aligned}$$

which is equivalent to the system

$$\begin{aligned} 0 &= \cos(t)A' + \sin(t)B' \\ tg(t) &= \cos(t)B' - \sin(t)A', \end{aligned}$$

Solving and noting that here $g(t) = \frac{1}{t}$, we obtain

$$A'(t) = -t\sin(t)g(t) \implies A(t) = -\int \sin(t)dt = \cos(t) + c_1.$$

and

$$B'(t) = t\cos(t)g(t) \implies B(t) = \int \cos(t)dt = \sin(t) + c_2.$$

Putting these into our ansatz, we've found

$$x_g(t) = c_1\frac{\cos(t)}{t} + c_2\frac{\sin(t)}{t} + \frac{1}{t}.$$

The initial conditions now yield

$$\begin{aligned} 0 = x(\pi) &= \frac{1 - c_1}{\pi} \\ 0 = x'(\pi) &= \frac{c_1 - 1}{\pi^2} + \frac{c_2}{\pi} \quad \implies \quad c_1 = 1,\ c_2 = 0, \end{aligned}$$

so that the solution to our IVP is

$$\boxed{x(t) = \frac{1 + \cos(t)}{t}}$$

Day 22: The Laplace transform; transform rules (3.1)

How can we systematically accommodate irregular sources?

Definition: The Laplace transform of a function $f(t)$ is the function $F(s) = \mathcal{L}\{f(t)\}(s)$ given by

$$F(s) = \int_0^\infty e^{-st} f(t) dt,$$

defined for all values of s for which the integral converges.

$\mathcal{L}$ is an operator on functions, both taking in and returning a function. It maps from the "t-domain" to the "s-domain".

It will be important that we can identify the Laplace transforms of simple functions.

Example: Find the Laplace transform of $f(t) = 1$.

$$\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st} dt = \lim_{B\to\infty} \int_0^B e^{-st} dt = \lim_{B\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^B = \frac{1}{s}\left[1 - \lim_{B\to\infty} e^{-sB}\right] = \begin{cases} \frac{1}{s} & s > 0 \\ \text{diverges} & s \le 0 \end{cases}$$

We therefore say $\mathcal{L}\{1\}(s) = \frac{1}{s}$ for $s > 0$.

□

Example: Find the Laplace transform of $f(t) = e^{(a+bi)t}$.

$$\begin{aligned} \mathcal{L}\{e^{(a+bi)t}\}(s) &= \int_0^\infty e^{-st} e^{(a+bi)t} dt = \int_0^\infty e^{-(s-a-bi)t} dt \\ &= \frac{1}{s-a-bi}\left[1 - \lim_{B\to\infty} e^{-(s-a-bi)B}\right] = \begin{cases} \frac{1}{s-a-bi} & s > a \\ \text{diverges} & s \le a \end{cases} \end{aligned}$$

Hence $\mathcal{L}\{e^{(a+bi)t}\}(s) = \frac{1}{s-a-bi}$ for $s > a$.

Equivalently, $\mathcal{L}\{e^{\alpha t}\}(s) = \frac{1}{s-\alpha}$ for $s > \mathrm{Re}(\alpha)$.

□

This is reflective of a more general property:

Theorem: If $F(s) = \mathcal{L}\{f\}(s)$ exists for $s > a$ and c is a real constant, then

$$\mathcal{L}\{e^{ct} f(t)\}(s) = F(s-c) \qquad \text{for } s > a + c. \tag{15}$$

Proof:

$$\mathcal{L}\{e^{ct} f(t)\}(s) = \int_0^\infty e^{-st} e^{ct} f(t) dt = \int_0^\infty e^{-(s-c)t} f(t) dt = F(s-c).$$

□

An exponential factor in the t-domain, then, corresponds to a translation in the s-domain.

Day 23: Laplace transform properties (3.1)

What properties of the transform can we leverage in computations?

As we have seen, linearity often leads to useful tools. The Laplace transform is linear:

Theorem: If $\mathcal{L}\{f_1\}$ and $\mathcal{L}\{f_2\}$ exist for $s > a_1$ and $s > a_2$ respectively, then

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\} \qquad \text{for } s > \max(a_1, a_2),$$

for any constants c_1, c_2.

Proof:

$$\begin{aligned}\mathcal{L}\{c_1 f_1 + c_2 f_2\}(s) &= \int_0^\infty e^{-st}\left[c_1 f_1(t) + c_2 f_2(t)\right] dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t) dt + c_2 \int_0^\infty e^{-st} f_2(t) dt \\ &= c_1\mathcal{L}\{f_1\}(s) + c_2\mathcal{L}\{f_2\}(s).\end{aligned}$$

□

Linearity allows us to extend results we know:

Example: Find the Laplace transform of $f(t) = \cos(at)$.

Using the identity $\cos(at) = \frac{1}{2}\left[e^{iat} + e^{-iat}\right]$, we find

$$\mathcal{L}\{\cos(at)\} = \frac{1}{2}\left[\mathcal{L}\{e^{iat}\} + \mathcal{L}\{e^{-iat}\}\right] = \frac{1}{2}\left[\frac{1}{s - ai} + \frac{1}{s + ai}\right] = \boxed{\frac{s}{s^2 + a^2}}$$

This holds for $s > 0$.

□

Above, citing the integral definition directly would have been significantly more tedious.

Another result aiding in the computation of Laplace transforms:

Theorem: If $F(s) = \mathcal{L}\{f(t)\}(s)$ exists for $s > a$, then

$$\mathcal{L}\{tf(t)\}(s) = -F'(s) \qquad \text{for } s > a.$$

Proof:

$$F'(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t) dt\right] = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt = \int_0^\infty (-t) e^{-st} f(t) dt = -\mathcal{L}\{tf(t)\}(s)$$

□

A factor of t in the t-domain, then, corresponds to differentiation in the s-domain.

One can iterate this result to obtain

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s). \tag{16}$$

Example: Find the Laplace transform of $f(t) = t^n$.

Applying the above result, we find

$$\mathcal{L}\{t^n\}(s) = (-1)^n \left(\frac{d}{ds}\right)^n [\mathcal{L}\{1\}(s)] = (-1)^n \left(\frac{d}{ds}\right)^n \left[\frac{1}{s}\right] = \boxed{\frac{n!}{s^{n+1}}}$$

□

Example: Find the Laplace transform of $f(t) = t^2 e^t + 2e^{-t}\cos(3t)$.

Using linearity and applying (15) to the results of previous examples, we have

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \mathcal{L}\{t^2\}(s-1) + 2\mathcal{L}\{\cos(3t)\}(s+1) \\ &= \frac{2}{(s-1)^3} + \frac{2(s+1)}{(s+1)^2+9}\end{aligned}$$

□

Among its most important properties is how the Laplace transform interacts with derivatives.

Theorem: If a differentiable $f(t)$ is of "exponential order" (see text) and $\mathcal{L}\{f'\}$ exists for $s > a$,

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0).$$

Proof:

$$\begin{aligned}\mathcal{L}\{f'\} &= \int_0^\infty e^{-st} f'(t)dt = \lim_{B\to\infty}\left[e^{-st}f(t)\Big|_0^B\right] + s\int_0^\infty e^{-st}f(t)dt \\ &= s\mathcal{L}\{f\} - f(0) + \lim_{B\to\infty}\left[e^{-sB}f(B)\right] \\ &= s\mathcal{L}\{f\} - f(0).\end{aligned}$$

□

Example: Find the Laplace transform of $f(t) = \sin(at)$.

We know from a previous example that $\mathcal{L}\{f'(t)\} = a\mathcal{L}\{\cos(at)\} = \frac{as}{s^2+a^2}$, so that

$$\frac{as}{s^2+a^2} = \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = s\mathcal{L}\{f(t)\}$$

Dividing by s, we've found

$$\boxed{\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}}$$

□

Observe that the conclusion of the above theorem can be iterated:

$$\mathcal{L}\{f''\} = s\mathcal{L}\{f'\} - f'(0) = s\left[s\mathcal{L}\{f\} - f(0)\right] - f'(0) = s^2\mathcal{L}\{f\} - sf(0) - f'(0),$$

and similarly

$$\mathcal{L}\{f'''\} = s\mathcal{L}\{f''\} - f''(0) = s^3\mathcal{L}\{f\} - s^2f(0) - sf'(0) - f''(0).$$

Continuing, one finds

Corollary:

$$\begin{aligned}\mathcal{L}\{f^{(n)}\} &= s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \\ &= s^n\mathcal{L}\{f\} - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0).\end{aligned}$$

□

Day 24: $\mathcal{L}^{-1}$ and solving IVP's with $\mathcal{L}$ (3.2)

How does the Laplace transform affect differential equations?

The most recent property makes Laplace transforms particularly well-suited to investigating IVP's.

Example: Find the Laplace transform $X(s) = \mathcal{L}\{x(t)\}(s)$ of the solution $x(t)$ to the IVP

$$x'' + x = t, \qquad x(0) = 0,\ x'(0) = 1$$

Taking the Laplace transform of both sides, we must have

$$s^2X(s) - s \cdot 0 - 1 + X(s) = \mathcal{L}\{t\}(s) = \frac{1}{s^2}.$$

Solving for $X(s)$, we find

$$\boxed{X(s) = \frac{1}{(s^2+1)s^2} + \frac{1}{s^2+1}}$$

To obtain the IVP solution $x(t)$, then, we would like to be able to invert the Laplace transform.

That is: knowing $X(s)$, we would like to make sense of

$$x(t) \stackrel{?}{=} \mathcal{L}^{-1}\{X(s)\}.$$

To see how this might be done, let us simplify:

$$\mathcal{L}\{x(t)\}(s) = X(s) = \frac{1+s^2}{(s^2+1)s^2} = \frac{1}{s^2} = \mathcal{L}\{t\}(s)$$

That is, we've found that $\mathcal{L}\{x(t)\} = \mathcal{L}\{t\}$.

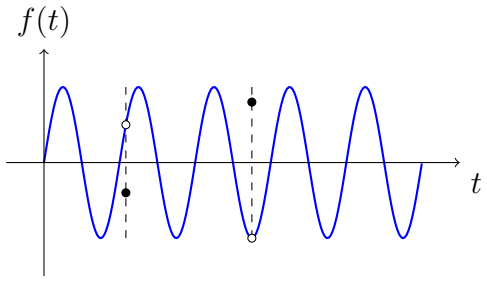
□

In the above, can we conclude that the solution to the IVP is $x(t) = t$? **Yes**:

Theorem: If $f(t)$ and $g(t)$ are piecewise continuous functions satisfying $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$ for $s > a$, then $f(t) = g(t)$ for all $t > 0$ at which both are continuous.

□

Example: The functions $f(t)$ (plotted below) and $g(t) = \sin(t)$ have the same Laplace transform.



□

If $\mathcal{L}\{f\}(s) = F(s)$, then, we may meaningfully call $f(t)$ the inverse Laplace transform of $F(s)$,

$$\mathcal{L}^{-1}\{F\} = f \qquad \Longleftrightarrow \qquad \mathcal{L}\{f\} = F,$$

up to this minor ambiguity.

While we must apparently understand $\mathcal{L}^{-1}$ to solve IVP's, we can't compute $\mathcal{L}^{-1}$ as directly as $\mathcal{L}$.

The best approach is generally to identify $\mathcal{L}^{-1}$ by inspection, comparing against known results:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{at}g(t)$	$G(s-a)$
$t^n g(t)$	$(-1)^n G^{(n)}(s)$
$g^{(n)}(t)$	$s^n G(s) - \sum_{k=1}^{n} s^{n-k} g^{(k-1)}(0)$

Linearity of $\mathcal{L}$ implies linearity of $\mathcal{L}^{-1}$, so inverse transforms may be taken term by term.

Example: Determine $f(t) = \mathcal{L}^{-1}\{F(s)\}$ for

$$F(s) = \frac{6s}{s^2-4} + \frac{s}{s^2-4s+5}.$$

Observing that

$$\frac{6s}{s^2-4} = \frac{6s}{(s-2)(s+2)} = \frac{3}{s-2} + \frac{3}{s+2} = 3\mathcal{L}\left\{e^{2t} + e^{-2t}\right\}$$

and

$$\frac{s}{s^2-4s+5} = \frac{s}{(s-2)^2+1} = \frac{s-2}{(s-2)^2+1} + \frac{2}{(s-2)^2+1} = \mathcal{L}\{e^{2t}\cos(t) + 2e^{2t}\sin(t)\},$$

we conclude that

$$\boxed{f(t) = \mathcal{L}^{-1}\left\{\frac{6s}{s^2-4} + \frac{s}{s^2-4s+5}\right\} = 3e^{-2t} + \left[3 + \cos(t) + 2\sin(t)\right]e^{2t}}$$

□

In solving IVP's, we often take $\mathcal{L}^{-1}$ of a rational function, requiring partial fraction decomposition.

Example: Determine $f(t) = \mathcal{L}^{-1}\{F(s)\}$ for

$$F(s) = \frac{1}{(s^2+1)s^2} + \frac{1}{s^2+1}.$$

We previously identified $f(t) = t$ by combining over a common denominator.

We got lucky: this is usually the wrong direction. More systematically, use PFD on the first term.

$$\frac{1}{(s^2+1)s^2} \stackrel{?}{=} \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2}.$$

We do not need to adjust the original $\frac{1}{s^2+1}$ term since we can already recognize it as $\mathcal{L}\{\sin(t)\}$.

We also cannot factor the denominators any further because they have no real roots. Proceeding,

$$\begin{aligned} 1 &\stackrel{?}{=} (As+B)s^2 + (Cs+D)(s^2+1) \\ &= (A+C)s^3 + (B+D)s^2 + Cs + D. \end{aligned}$$

Noting the LHS has no s^3, s^2, or s terms, we obtain four equations:

$$\begin{aligned} 0 &= A + C, \\ 0 &= B + D, \\ 0 &= C, \\ 1 &= D, \end{aligned}$$

which together imply $A = 0$, $B = -1$, $C = 0$, $D = 1$, so we have found

$$\frac{1}{(s^2+1)s^2} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

Putting everything together, we've found

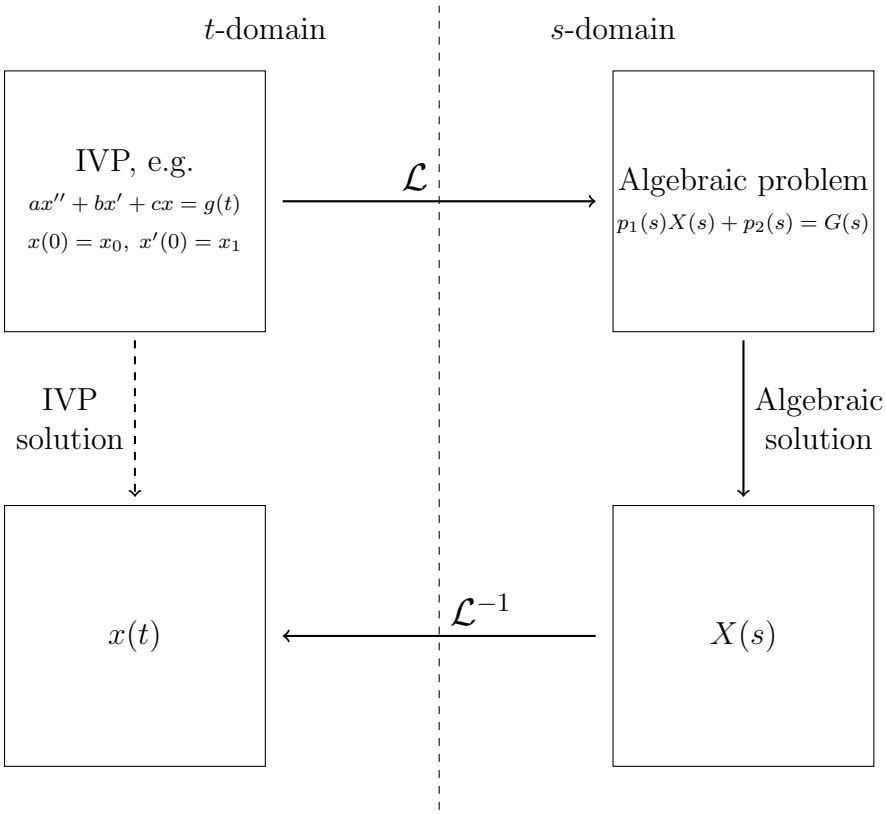
$$F(s) = \frac{1}{s^2} = \mathcal{L}\{t\},$$

so that $\boxed{f(t) = t}$

□

Day 25: Transforms of discontinuous functions (3.1, 3.2)

With an understanding of $\mathcal{L}$ and $\mathcal{L}^{-1}$, the following pipeline for solving IVP's has emerged:



Example: Solve the IVP

$$x'' - 4x' + 8x = 16t, \qquad x(0) = 1,\ x'(0) = 0$$

Applying $\mathcal{L}$, we have

$$\begin{aligned}
s^2X(s) - s - 4sX(s) + 4 + 8X(s) &= \frac{16}{s^2} \\
(s^2 - 4s + 8)X(s) - s + 4 &= \frac{16}{s^2} \\
\implies \quad ((s-2)^2 + 4)X(s) &= \frac{16}{s^2} + s - 4 \\
\implies \quad X(s) &= \frac{16}{((s-2)^2+4)s^2} + \frac{s-4}{(s-2)^2+4} \\
&= \frac{2-s}{(s-2)^2+4} + \frac{s+2}{s^2} + \frac{s-4}{(s-2)^2+4} \\
&= -\frac{2}{(s-2)^2+4} + \frac{1}{s} + \frac{2}{s^2}.
\end{aligned}$$

Applying $\mathcal{L}^{-1}$, we deduce

$$\boxed{x(t) = -e^{2t}\sin(2t) + 2t + 1}$$

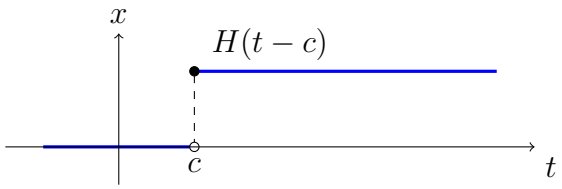
□

How can we formally study and incorporate discontinuous and piecewise functions?

Efficient manipulation of piecewise functions will use the unit step function, or Heaviside function,

$$H(t) := \begin{cases} 0 & t < 1 \\ 1 & t \geq 1. \end{cases}$$

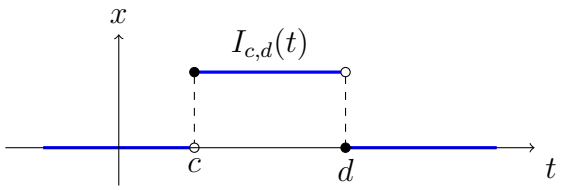
This encodes behavior switching on at $t = 0$. The translation $H(t - c)$ switches on at $t = c$.



We can then construct the indicator function $I_{c,d}(t)$ for $d > c$,

$$I_{c,d}(t) := H(t - c) - H(t - d) = \begin{cases} 0 & t < c \text{ or } t \geq d \\ 1 & c \leq t < d, \end{cases}$$

which encodes behavior present only in the range $[c, d)$.



Example: The piecewise function

$$f(t) = \begin{cases} 0 & t < 1 \\ t & 1 \leq t < 2 \\ e^t & 2 \leq t < 3 \\ 2 & 3 \leq t \end{cases}$$

can be succinctly expressed as

$$f(t) = tI_{1,2}(t) + e^t I_{2,3}(t) + 2I_3(t).$$

This can also be written

$$\begin{aligned} f(t) &= t\left[H(t-1) - H(t-2)\right] + e^t\left[H(t-2) - H(t-3)\right] + 2H(t-3) \\ &= tH(t-1) + (e^t - t)H(t-2) + (2 - e^t)H(t-3). \end{aligned}$$

□

One may readily compute

$$\mathcal{L}\{H(t-c)\} = \int_0^\infty e^{-ts} H(t-c)dt = \int_c^\infty e^{-ts}dt = \frac{e^{-cs}}{s}, \quad s > 0.$$

To use piecewise functions, we are interested in the transform of a product with $H(t-c)$:

Theorem If $c \geq 0$ and $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$, then

$$\mathcal{L}\{f(t-c)H(t-c)\} = e^{-cs}F(s), \quad s > a \tag{17}$$

Equivalently,

$$\boxed{\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)H(t-c)} \tag{18}$$

Proof: Evaluating,

$$\begin{aligned}\mathcal{L}\{f(t-c)H(t-c)\} &= \int_0^\infty e^{-st}f(t-c)H(t-c)dt = \int_c^\infty e^{-st}f(t-c) \\ &= \int_0^\infty e^{-s(\tau+c)}f(\tau)d\tau = e^{-sc}\int_0^\infty e^{-s\tau}f(\tau)d\tau \\ &= e^{-sc}F(s),\end{aligned}$$

□

Translation in the t-domain, then, yields an exponential factor in the s-domain. Compare with (15).

While (18) is useful in applying $\mathcal{L}^{-1}$, when applying $\mathcal{L}$ it is convenient to recast (17) as

$$\boxed{\mathcal{L}\{f(t)H(t-c)\} = e^{-cs}\mathcal{L}\{f(t+c)\}} \tag{19}$$

Example: Find the Laplace transform of the piecewise function $f(t)$ from the previous example.

We found above that we can express

$$\begin{aligned}f(t) &= tH(t-1) + (e^t - t)H(t-2) + (2-e^t)H(t-3) \\ &= f_1(t)H(t-1) + f_2(t)H(t-2) + f_3(t)H(t-3)\end{aligned}$$

Noting that

$$\begin{aligned}\mathcal{L}\{f_1(t+1)\} = \mathcal{L}\{t+1\} &= \frac{1}{s^2} + \frac{1}{s} \\ \mathcal{L}\{f_2(t+2)\} = \mathcal{L}\{e^{t+2} - t - 2\} &= \frac{e^2}{s-1} - \frac{1}{s^2} - \frac{2}{s} \\ \mathcal{L}\{f_3(t+3)\} = \mathcal{L}\{2 - e^{t+3}\} &= \frac{2}{s} - \frac{e^3}{s-1},\end{aligned}$$

we have

$$\boxed{\mathcal{L}\{f(t)\} = \frac{e^{-s} - 2e^{-2s} + 2e^{-3s}}{s} + \frac{e^{-s} - e^{-2s}}{s^2} + \frac{e^{2-2s} - e^{3-3s}}{s-1}}$$

□

Day 26: IVP's with discontinuous functions (3.2)

How can we formally study and incorporate discontinuous and piecewise functions?

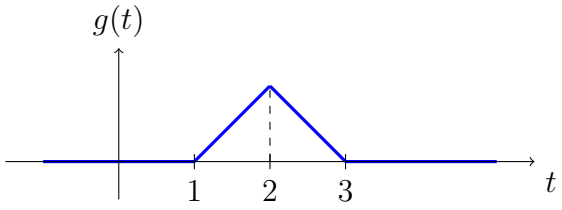
Example: A mass of 1 kg is sitting at rest while affixed to a vertical spring of stiffness $k = 50$ N/m and damping coefficient $\gamma = 2\ \frac{\text{N}}{\text{m/s}}$. Beginning at $t = 1$ s, the spring's support is raised at 1 m/s for one second, and then immediately lowered (at the same speed) back to its original position. Find the mass' position $x(t)$ (in meters) as a function of time.

Measuring $x(t)$ from the original equilibrium, it is governed by the IVP

$$x'' + 2x' + 50x = 50g(t), \qquad x(0) = 0,\ x'(0) = 0,$$

where $g(t)$ is the position function of the spring's support, a triangular pulse given by

$$\begin{aligned} g(t) &= (t-1)I_{1,2}(t) - (t-3)I_{2,3}(t) \\ &= (t-1)H(t-1) - 2(t-2)H(t-2) + (t-3)H(t-3) \end{aligned}$$



Applying $\mathcal{L}$ to our IVP using (19) three times, we find

$$\begin{aligned} (s^2 + 2s + 50)X(s) &= 50G(s) \\ ((s+1)^2 + 49)X(s) &= 50\frac{e^{-s} - 2e^{-2s} + e^{-3s}}{s^2} \\ \implies \qquad X(s) &= 50\frac{e^{-s} - 2e^{-2s} + e^{-3s}}{s^2((s+1)^2 + 49)} \\ &= \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{25}\left[\frac{s-23}{(s+1)^2+49} + \frac{25-s}{s^2}\right] \\ &= \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{25}\left[\frac{s+1-24}{(s+1)^2+49} + \frac{25}{s^2} - \frac{1}{s}\right] \\ &= \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{25}\mathcal{L}\left\{e^{-t}\left(\cos(7t) - \frac{24}{7}\sin(7t)\right) + 25t - 1\right\} \end{aligned}$$

Denoting by $f(t)$ the transformed function in the last line,

$$f(t) = \frac{1}{25}\left[e^{-t}\left(\cos(7t) - \frac{24}{7}\sin(7t)\right) + 25t - 1\right],$$

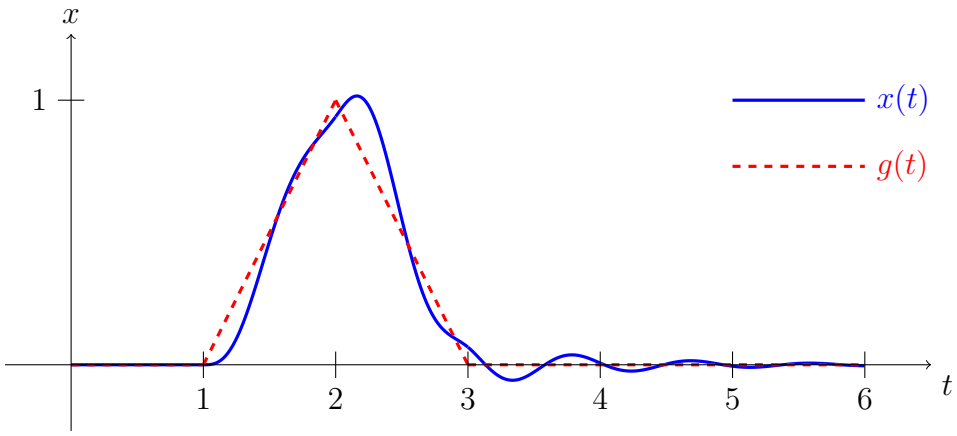
and its transform by $F(s)$, we've found that

$$X(s) = (e^{-s} - 2e^{-2s} + e^{-3s})F(s)$$

Applying $\mathcal{L}^{-1}$ and using (18), we conclude

$$\boxed{x(t) = f(t-1)H(t-1) - 2f(t-2)H(t-2) + f(t-3)H(t-3)}$$

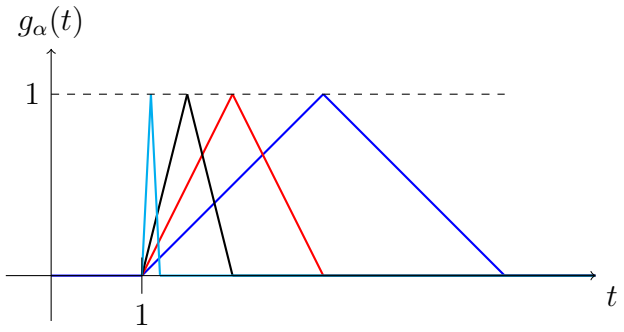
This result is plotted below.



□

One can see how this response profile varies if the movement of the support by 1 m is slowed or quickened by varying the parameter α, the time taken to raise/lower the mass, in this Desmos plot.

$$g_\alpha(t) = \begin{cases} (t-1)/\alpha & 1 \le t \le 1+\alpha \\ (2\alpha-(t-1))/\alpha & 1+\alpha \le t \le 1+2\alpha \\ 0 & \text{otherwise} \end{cases}$$



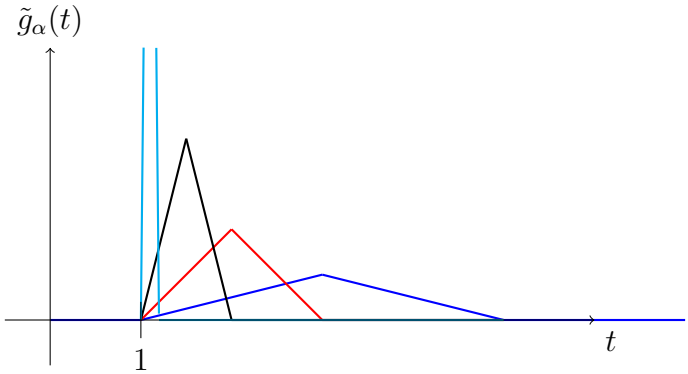
Observe in the demo that something akin to resonance occurs around $\alpha \sim 0.4$, when the timescale of raising and lowering the support is comparable to the natural period of the spring.

Further, the mass has essentially no response if the support moves up and down too quickly ($\alpha \to 0$).

These pulses had a conserved magnitude: the mass doesn't respond to a quick, fixed-amplitude jolt.

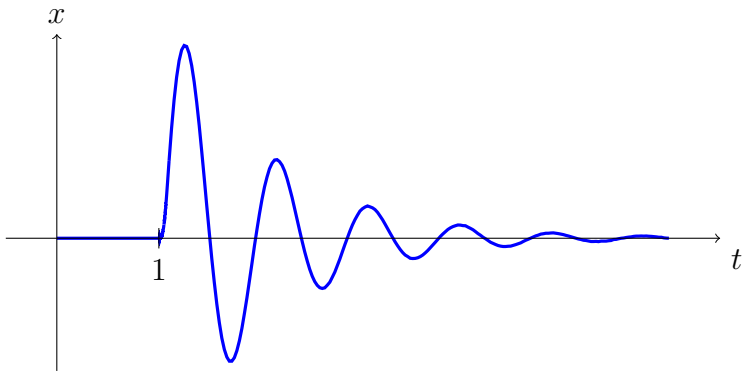
What if the pulse's magnitude is scaled in inverse proportion to its temporal extent α?

$$\tilde{g}_\alpha(t) = \begin{cases} (t-1)/\alpha^2 & 0 \le t \le \alpha \\ (2\alpha-(t-1))/\alpha^2 & \alpha \le t \le 2\alpha \\ 0 & \text{otherwise} \end{cases}$$



One can see the behavior of the response to this new family of pulses in this Desmos plot.

There is now a non-zero limiting response, plotted below, as $\alpha \to 0$ (to a very sharp, tall spike).



How should this be interpreted? The new family of pulses now has conserved *area*: all satisfy

$$\int_{-\infty}^{\infty} \tilde{g}_\alpha(t)dt = \int_{1}^{1+2\alpha} \tilde{g}_\alpha(t)dt = 1.$$

In the spring setup, $50\tilde{g}_\alpha(t)$ was the effective external driving force. Since the integral of force over time is transferred *momentum*, or impulse, this new family of pulses all impart the same momentum.

Taking $\alpha \to 0$ models a sudden and rapid impulse, e.g. the strike of a hammer or a sharp collision.

It is often of great interest to know how a system responds to such processes, its impulse response. We will study how this can be assessed formally next time.

Hit play on the t slider in this Desmos demo to visualize the response to these pulses. Switch from the g_α to the $\tilde{g}_\alpha$ pulses by adjusting the n slider from $n = 1$ to $n = 2$.

Day 27: Laplace with impulses (3.4)

How can we formally study and incorporate impulses?

Solving with $\tilde{g}_\alpha$ and taking $\alpha \to 0$ is tedious: vastly simplified by the dirac delta "function" $\delta(t)$.

Definition: The Dirac delta function, or unit impulse function $\delta(t-t_0)$ is defined to satisfy

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0) \tag{20}$$

for every function $f(t)$ continuous around t_0.

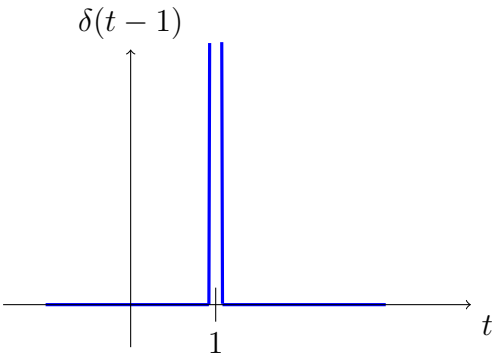
□

$\delta(t)$ is not literally a function: there does not exist a function with the above property. Instead, it is an idealized and formal mathematical object entirely characterized by (20).

Nonetheless, $\delta(t)$ captures the limiting behavior of function families like $\tilde{g}_\alpha$: one can show

$$\begin{aligned}\lim_{\alpha\to 0}\left[\int_{-\infty}^{\infty} f(t)\tilde{g}_\alpha(t)dt\right] &= \lim_{\alpha\to 0}\left[\int_{1}^{1+2\alpha} f(t)\tilde{g}_\alpha(t)dt\right] \\ &= f(1)\cdot\lim_{\alpha\to 0}\left[\int_{1}^{1+2\alpha} \tilde{g}_\alpha(t)dt\right] \\ &= f(1).\end{aligned}$$

For this reason, $\delta(t-1)$ is heuristically visualized like $\tilde{g}_\alpha$ as $\alpha \to 0$, an infinite spike at $t=1$:



While not technically correct, this is a useful visual to have in mind.

A variety of properties can be demonstrated from (20): replacing $f(t)$ with $I_{a,b}(t)f(t)$ yields

$$\int_a^b f(t)\delta(t-t_0)dt = \begin{cases} f(t_0) & a < t_0 < b \\ 0 & t_0 < a \text{ or } t_0 > b. \end{cases}$$

Taking $f(t)=1$, we see that the "area" of the delta function's "spike" is 1, like $\tilde{g}_\alpha$.

Similarly, we may compute the Laplace transform

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0) = e^{-st_0}$$

for $t_0 > 0$. We can now use δ to model the response of physical systems to rapid and sudden impulses.

Example: Derive the impulse response plotted at the end of Day 26.

The response there was obtained by solving the IVP

$$x'' + 2x' + 50x = 50\tilde{g}_\alpha(t), \qquad x(0) = 0,\ x'(0) = 0,$$

and letting $\alpha \to 0$. We can do this *much* more efficiently by replacing $\tilde{g}_\alpha(t)$ with $\delta(t-1)$:

$$x'' + 2x' + 50x = 50\delta(t-1), \qquad x(0) = 0,\ x'(0) = 0.$$

Physically, this models the response to a hammer strike imparting 50 kg·m/s of momentum at $t = 1$.

Applying $\mathcal{L}$, we obtain

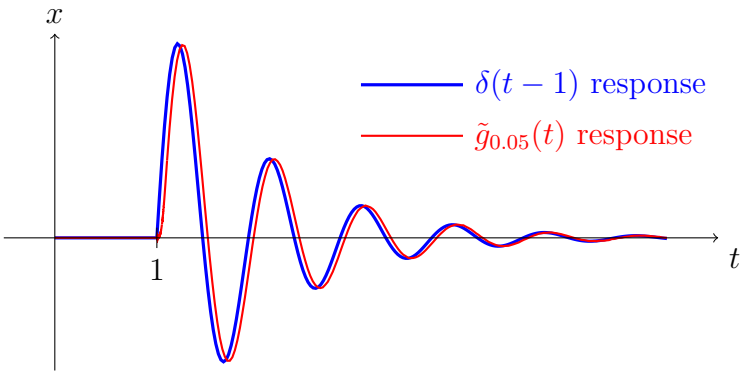
$$\begin{aligned}(s^2 + 2s + 50)X(s) &= 50e^{-s} \\ \implies \qquad X(s) &= \frac{50}{s^2+2s+50} = \frac{50e^{-s}}{(s+1)^2+49},\end{aligned}$$

and denoting

$$F(s) := \frac{1}{(s+1)^2+49}, \qquad f(t) := \mathcal{L}^{-1}\{F(s)\} = \frac{1}{7}e^{-t}\sin(7t),$$

we see that $x(t) = 50f(t-1)H(t-1)$, or

$$\boxed{x(t) = \frac{50}{7}e^{-(t-1)}\sin(7(t-1))H(t-1)}$$



□

The text uses a different family of functions than $\tilde{g}_\alpha$ to model δ in a limit. Both models include extraneous details irrelevant to the impulse response, and these are idealized away by δ.

The delta function is a ubiquitous tool for encoding phenomena at a single point in space or time.

Example: The mass-spring system from the prior example hangs, initially at rest, from a table which is struck with a hammer driving a nail once per second starting at $t = 1$, delivering 1 kg·m/s of momentum to the mass each time. Find the mass' position $x(t)$.

This scenario is described by the IVP

$$x'' + 2x' + 50x = \sum_{k=1}^{\infty} \delta(t-k), \qquad x(0) = 0,\ x'(0) = 0.$$

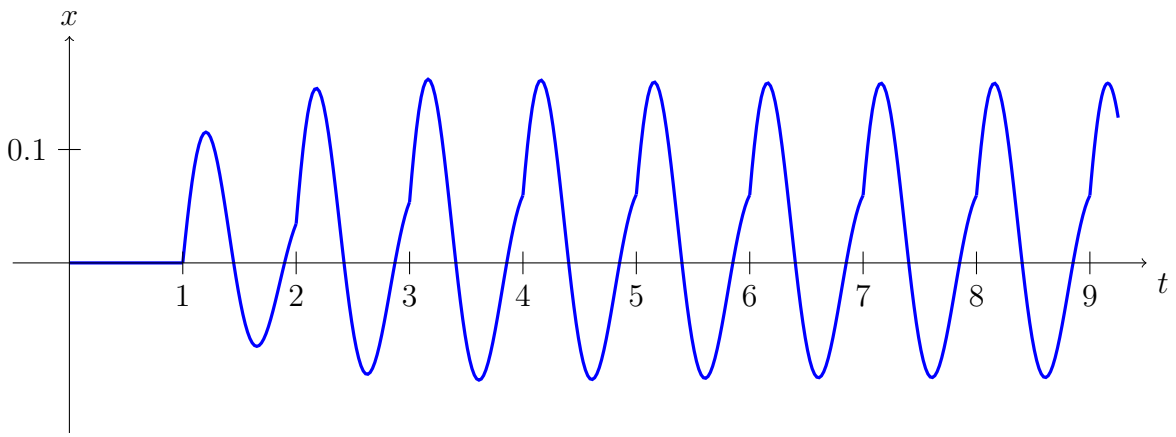
Applying $\mathcal{L}$ (and denoting $F(s)$ and $f(t)$ as above) yields

$$(s^2 + 2s + 50)X(s) = \sum_{k=1}^{\infty} e^{-ks}$$

$$\implies \qquad X(s) = \sum_{k=1}^{\infty} e^{-ks} F(s),$$

so that

$$\boxed{x(t) = \sum_{k=1}^{\infty} f(t-k)H(t-k)}$$



□

One can see how the response profile varies with the time T between hammer strikes by varying the slider in this Desmos demo.

Observe that resonance occurs around $T \sim 0.9$, near the natural period of the spring $\frac{2\pi}{\sqrt{50}} \approx 0.889$.

Hit play on the t slider in this Desmos demo to visualize this response.

Final Laplace trivia

Recall that the Laplace transform is defined by

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^{\infty} e^{-st} f(t)\,dt,$$

so long as this integral converges (absolutely).

Since the decay e^{-st} is stronger for larger s: if $F(s)$ exists for $s = a$, then it also exists for $s > a$. Moreover,

$$\lim_{s\to\infty} F(s) = 0$$

for "well-behaved" $f(t)$ (e.g., of exponential order).

In particular, if both $f(t)$ and $f'(t)$ are of exponential order,

$$\begin{aligned} 0 = \lim_{s\to\infty} \mathcal{L}\{f'(t)\} &= \lim_{s\to\infty} [s \cdot F(s) - f(0)] \\ \implies \quad f(0) &= \lim_{s\to\infty} s \cdot F(s) \end{aligned}$$

Example: If $f(t) = \sin(at)$ and $g(t) = \cos(at)$, then

$$\begin{aligned} 0 = f(0) = \lim_{s\to\infty} s \cdot F(s) &= \lim_{s\to\infty} \left[\frac{as}{s^2 + a^2}\right] \\ 1 = g(0) = \lim_{s\to\infty} s \cdot G(s) &= \lim_{s\to\infty} \left[\frac{s^2}{s^2 + a^2}\right] \end{aligned}$$

□

Day 28: Intro to 2D Systems of ODEs (4.1,6.3)

How do we formulate and visualize systems of ODE's?

A single first-order ODE (one-dimensional) for one unknown $x(t)$ has the general form

$$x' = f(t, x).$$

A system of two first-order ODE's (two-dimensional) in two unknowns $x(t)$, $y(t)$ has the form

$$\begin{aligned} x' &= f_1(t, x, y) \\ y' &= f_2(t, x, y), \end{aligned} \tag{21}$$

Example: Show that $x(t) = e^{-t}\cos(t)$, $y(t) = e^{-t}\sin(t)$ together solve the system

$$\begin{aligned} x' &= -x - y \\ y' &= x - y. \end{aligned}$$

Computing derivatives, we indeed find

$$\begin{aligned} x'(t) &= -e^{-t}\cos(t) - e^{-t}\sin(t) = -x(t) - y(t) \\ y'(t) &= e^{-t}\sin(t) - e^{-t}\cos(t) = x(t) - y(t). \end{aligned}$$

□

We can again approximate solutions to (21) with Euler's method. We divide (t_0, T) into N subintervals between $t_0, t_1, \ldots, t_N$, estimating both $x(t_n) \approx X_n$ and $y(t_n) \approx Y_n$ recursively in tandem:

$$\boxed{\begin{aligned} X_{n+1} &= X_n + f_1(t_n, X_n, Y_n)\Delta t \\ Y_{n+1} &= Y_n + f_2(t_n, X_n, Y_n)\Delta t \end{aligned}}$$

A well-posed IVP requires pairing (21) with the initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$.

Example: Use Euler's method with $N = 4$ steps to approximate $x(t)$ and $y(t)$ at $t = 2$ in the IVP

$$\begin{aligned} x' &= -x - y, \qquad x(0) = 1, \\ y' &= x - y, \qquad y(0) = 0. \end{aligned}$$

The range from $t_0 = 0$ to $t_4 = 2$ is divided into 4 segments of width $\Delta t = \frac{1}{2}$. We have $X_0 = 1$, $Y_0 = 0$, $f_1(t, x, y) = -x - y$, and $f_2(t, x, y) = x - y$. We may proceed in a table as before:

n	t_n	X_n	Y_n	$f_1(t_n, X_n, Y_n)$	$f_2(t_n, X_n, Y_n)$	ΔX	ΔY
0	0	1	0	-1	1	-0.5	0.5
1	0.5	0.5	0.5	-1	0	-0.5	0
2	1	0	0.5	-0.5	-0.5	-0.25	-0.25
3	1.5	-0.25	0.25	0	-0.5	0	-0.25
4	2	-0.25	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$

So we've found the estimates $x(2) \approx X_4 = -0.25$ and $y(2) \approx Y_4 = 0$.

One can see the Euler's method approximation converge to the true solution in this Desmos demo.

□

(21) is generally a prohibitively difficult problem. To proceed, we restrict to the linear case:

$$\begin{aligned} x' &= p_{11}(t)x + p_{12}(t)y + g_1(t) \\ y' &= p_{21}(t)x + p_{22}(t)y + g_2(t), \end{aligned} \tag{22}$$

This system is homogeneous if $g_1(t) = g_2(t) = 0$ and autonomous if $p_{ij}(t)$ and $g_i(t)$ are constants. Less restrictively, we say it has constant coefficients if only the $p_{ij}(t)$ are constants.

We can once again broadly characterize when this problem admits a solution:

Theorem: Suppose that each of $g_i(t)$ and $p_{ij}(t)$ is continuous on an interval $I = (\alpha, \beta)$ containing t_0. Then there exists a unique pair $x(t)$, $y(t)$ solving the IVP

$$\begin{aligned} x' &= p_{11}(t)x + p_{12}(t)y + g_1(t), \qquad x(0) = x_0 \\ y' &= p_{21}(t)x + p_{22}(t)y + g_2(t), \qquad y(0) = y_0 \end{aligned}$$

on I for any choice of initial values x_0, y_0.

□

Even so, constructing these solutions to (22) is no easy feat for a general linear system.

Both qualitative and analytical techniques begin with the homogeneous and autonomous case:

$$\begin{aligned} x' &= ax + by, \qquad x(0) = x_0 \\ y' &= cx + dy, \qquad y(0) = y_0, \end{aligned}$$

where a, b, c, d are constants. We are primarily concerned with this system for the time being.

Linear system IVP's with constant coefficients are amenable to solution via the Laplace transform.

Example: Construct the solution to the IVP

$$\begin{aligned} x' &= -x - y, \qquad x(0) = 1, \\ y' &= x - y, \qquad y(0) = 0. \end{aligned}$$

Applying $\mathcal{L}$ to both equations, we obtain

$$\begin{aligned} sX(s) - 1 &= -X(s) - Y(s) \\ sY(s) - 0 &= X(s) - Y(s), \end{aligned}$$

and grouping $X(s)$ and $Y(s)$ terms yields

$$\begin{aligned} (s+1)X(s) + Y(s) &= 1 \\ X(s) - (s+1)Y(s) &= 0. \end{aligned}$$

This has turned the differential t problem into an algebraic s problem. Solving for $X(s)$ and $Y(s)$,

$$X(s) = \frac{s+1}{(s+1)^2+1} = \mathcal{L}\{e^{-t}\cos(t)\}$$
$$Y(s) = \frac{1}{(s+1)^2+1} = \mathcal{L}\{e^{-t}\sin(t)\},$$

so that we've found

$$\boxed{\begin{aligned} x(t) &= e^{-t}\cos(t) \\ y(t) &= e^{-t}\sin(t) \end{aligned}}$$

□

Example: The Coriolis effect is an apparent deviation from Newton's laws due to Earth's rotation. Because of this, the east/north components of a fast horizontally-moving object's velocity satisfy

$$\begin{aligned} v_e' &= \quad 2\omega\sin(\varphi)v_n, \\ v_n' &= -2\omega\sin(\varphi)v_e \end{aligned} \tag{23}$$

at latitude φ, where $\omega \approx \frac{2\pi}{24 \text{ hrs}}$ is the Earth's angular velocity.

With linear air resistance included, this becomes

$$\begin{aligned} v_e' &= -\gamma v_e + 2\omega\sin(\varphi)v_n, \\ v_n' &= -2\omega\sin(\varphi)v_e - \gamma v_n \end{aligned} \tag{24}$$

These systems are linear, autonomous, and homogeneous.

□

In one dimension, we qualitatively understood autonomous equations

$$x' = f(x)$$

via the phase line, the x-axis adorned with directed arrows.

In two dimensions, we can similarly understand autonomous (sometimes called dynamical) systems

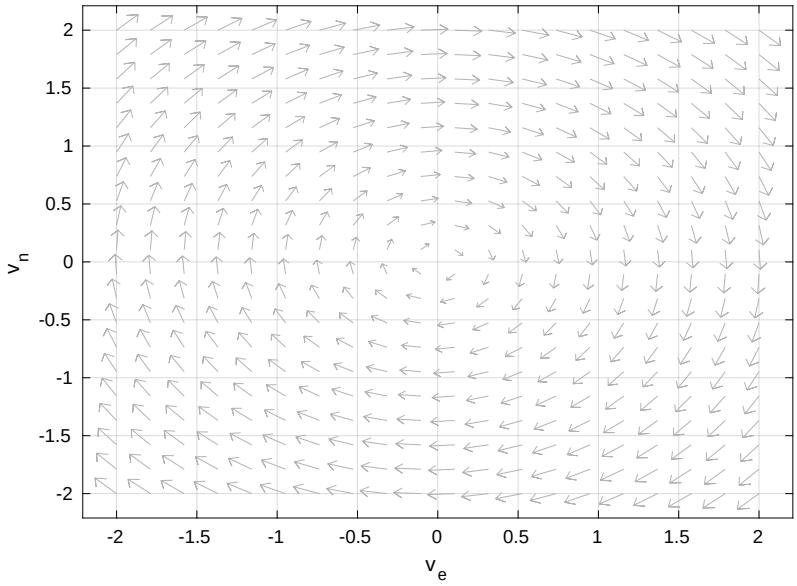
$$\begin{aligned} x' &= f_1(x,y) \\ y' &= f_2(x,y), \end{aligned}$$

via the phase plane, the x-y plane adorned with directed arrows and/or sample trajectories.

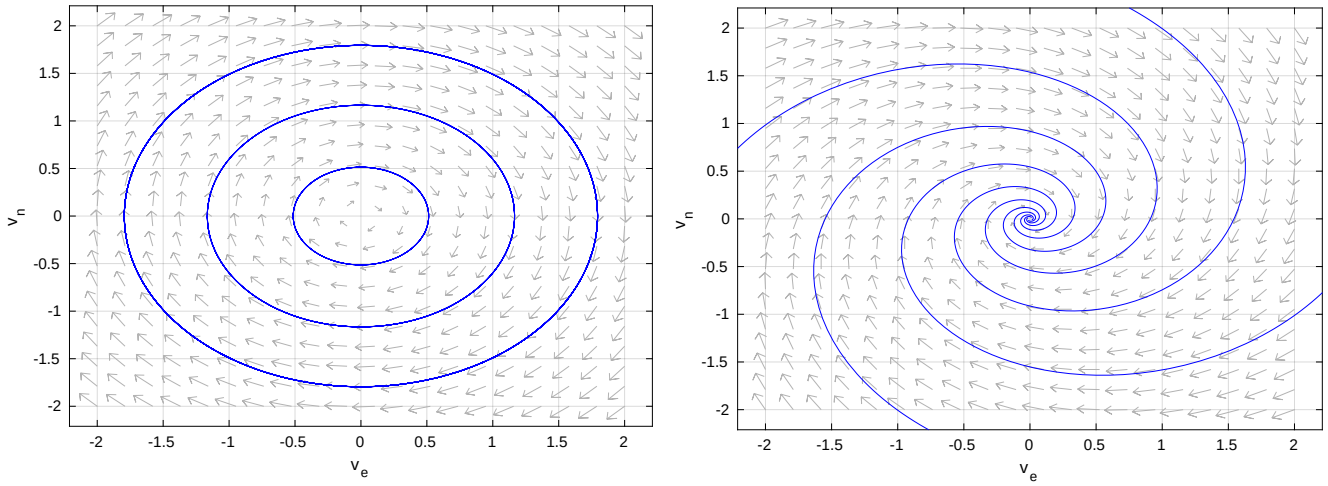
A plot of arrows indicating the directions followed by solutions in the phase plane is a direction field.

Such a plot also containing a representative sample of solution trajectories is a phase portrait.

Example: The direction field for the system (23):



Phase portraits for the systems (23) (left) and (24) (right):



On the right, the Coriolis terms are immensely exaggerated compared to realistic damping for the purpose of illustration. The circular nature of these trajectories illustrates that the Coriolis effect turns projectiles to the right in the Norther hemisphere ($\varphi > 0$, plotted above).

□

As in the 1D case, it is essential to qualitative understanding to assess stability of critical points (or equilibria): pairs (x, y) at which $x' = y' = 0$, i.e. solutions to

$$0 = f_1(x, y)$$
$$0 = f_2(x, y).$$

In the linear, constant-coefficient case, the unique critical point is generically the origin, $(x, y) = (0, 0)$. Note that phase portrait structure above is centered around this point.

Day 29: Order versus dimension: D operator elimination (4.1)

How are order and dimension related?

We have spent a fair bit of time understanding second-order linear ODE's,

$$ax'' + bx' + cx = g(t).$$

Recall that we have seen how order can always be traded for dimension.

Example: The capacitor's charge $q(t)$ in an RLC circuit is governed by the second-order ODE

$$Lq'' + Rq' + \frac{q}{C} = \mathcal{E}(t).$$

This is equivalent to the first-order 2D system in $x(t) := q(t)$ and $y(t) := q'(t)$ given by

$$\begin{aligned} x' &= y \\ y' &= -\frac{1}{LC}x - \frac{R}{L}y + \frac{\mathcal{E}(t)}{L}, \end{aligned}$$

□

The reverse is also true in the linear case: 2D linear systems can be recast as second-order ODE's.

To see this, it is illuminating to consider the action of the derivative as an operator, a transformation of an input function $x(t)$ to the output function $x'(t)$. We notate this by juxtaposition by D:

$$\begin{aligned} Dx &:= x', \text{ or} \\ (Dx)(t) &= x'(t). \end{aligned}$$

We may similarly consider multiplication by a particular function $p(t)$ to be an operator which transforms an input $x(t)$ into the output $p(t)x(t)$, indicated by juxtaposition by p:

$$(px)(t) := p(t)x(t).$$

Both of these operators are linear:

Definition: An operator on functions O is called linear if

$$O[c_1x_1 + c_2x_2] = c_1O[x_1] + c_2O[x_2].$$

□

The operators D and p may be combined to form a collection of differential operators.

We can extract second-order equations for $x(t)$ and $y(t)$ in a 2D system by doing algebra with D.

Example: Find second-order equations satisfied by solutions $x(t)$ and $y(t)$ to the system

$$\begin{aligned} x' &= 2x + 2y + t \\ y' &= x + 3y - \cos(t). \end{aligned}$$

Writing these in terms of D and grouping x terms on the left and y terms on the right, we have

$$\begin{aligned}(D-2)x &= 2y + t \\ x &= (D-3)y + \cos(t).\end{aligned} \tag{25}$$

We may eliminate y by applying $(D-3)$ to the first equation and 2 to the second, obtaining

$$\begin{aligned}(D-3)(D-2)x = 2(D-3)y + (D-3)t &= 2(D-3)y + 1 - 3t \\ 2x &= 2(D-3)y + 2\cos(t),\end{aligned}$$

and subtracting to find

$$\begin{aligned}[(D-3)(D-2) - 2]\, x &= 1 - 3t - 2\cos(t) \\ (D^2 - 5D + 4)x &= 1 - 3t - 2\cos(t) \\ \implies \qquad \boxed{x'' - 5x' + 4x = 1 - 3t - 2\cos(t)}&\end{aligned}$$

Returning to (25), we may eliminate x by applying $(D-2)$ to the second equation,

$$\begin{aligned}(D-2)x &= 2y + t \\ (D-2)x &= (D-2)(D-3)y + (D-2)\cos(t) = (D-2)(D-3)y - \sin(t) - 2\cos(t),\end{aligned}$$

and subtracting to find

$$\begin{aligned}0 &= [(D-2)(D-3) - 2]\, y - \sin(t) - 2\cos(t) - t \\ \implies \quad (D^2 - 5D + 4)y &= \sin(t) + 2\cos(t) + t \\ \implies \qquad \boxed{y'' - 5y' + 4y = \sin(t) + 2\cos(t) + t}&\end{aligned}$$

□

Note: the equations for x and y above have the same homogeneous part. This is no coincidence.

Day 30: D operator elimination (cont'd); systems of 2 linear algebraic equations (4.2)

We work through the elimination in the general homogeneous, autonomous case,

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned} \tag{26}$$

In terms of D,

$$\begin{aligned} (D-a)x &= by \\ cx &= (D-d)y. \end{aligned}$$

After applying $(D-d)$ to the first equation and b to the second, subtracting eliminates y.

After applying c to the first equation and $(D-a)$ to the second, subtracting eliminates x.

$$\begin{aligned} 0 &= [(D-d)(D-a) - bc]\, x = (D^2 - (a+d)D + ad - bc)x = x'' - (a+d)x' + (ad-bc)x \\ 0 &= [(D-a)(D-d) - bc]\, y = (D^2 - (a+d)D + ad - bc)y = y'' - (a+d)y' + (ad-bc)y \end{aligned}$$

Hence, the homogeneous x and y equations are the same in this case, with characteristic equation

$$\boxed{\lambda^2 - (a+d)\lambda + ad - bc = 0}$$

We also call this the characteristic equation of the system (26), and its roots are (26)'s eigenvalues.

A word of caution: the above conclusion does not hold with non-constant coefficients. In that scenario, the x and y equations will be different.

Example: Find second-order equations satisfied by solutions $x(t)$ and $y(t)$ to the system

$$\begin{aligned} x' &= x - ty \\ y' &= x + 2y. \end{aligned}$$

In terms of D, this reads

$$\begin{aligned} (D-1)x &= -ty \\ x &= (D-2)y. \end{aligned} \tag{27}$$

Applying $(D-1)$ to the second equation and subtracting, we find

$$0 = [(D-1)(D-2) + t]\, y = (D^2 - 3D + 2 + t)y = y'' - 3y' + (t+2)y.$$

On the other hand, applying $(D-2)$ to the first equation and t to the second, we have

$$\begin{aligned} (D-2)(D-1)x &= -(D-2)(ty) \\ tx &= t(D-2)y. \end{aligned}$$

In adding these, however, the RHS does not cancel, so x does not satisfy the same equation as y:

$$[(D-2)(D-1) + t]\, x = t(D-2)y - (D-2)(ty) = tDy - D(ty) = -y.$$

To fully eliminate y, multiply this result by t and subtract the first line of (27) to obtain

$$0 = \big[t(D-2)(D-1)x + t^2 - (D-1)x\big] = (tD^2 - (3t+1)D + t^2 + 2t + 1)x = tx'' - (3t+1)x' + (t^2+2t+1)x.$$

Hence, x and y satisfy the two distinct equations

$$\boxed{\begin{aligned} tx'' - (3t+1)x' + (t^2+2t+1)x &= 0 \\ y'' - 3y' + (t+2)y &= 0. \end{aligned}}$$

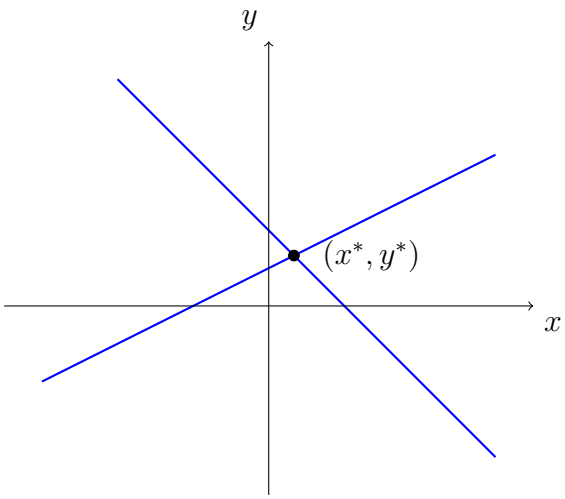
□

Brief aside on formalizing algebraic systems of equations

A general system of linear algebraic equations for the unknowns x, y has the form

$$\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta \end{aligned} \tag{28}$$

Geometrically, each equation characterizes a line in the x-y plane, and solutions (x^*, y^*) of the system are points of intersection:



A unique solution exists so long as the lines are not parallel, which is equivalent to the constraint

$$\Delta := ad - bc \neq 0.$$

Indeed, the lines have slopes $-\dfrac{a}{b}$ and $-\dfrac{c}{d}$, and setting these equal would imply $\Delta = 0$.

It will be conceptually advantageous to cast such equations in matrix and vector notation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Note that a matrix acts on (or "multiplies") a vector to produce a new vector, e.g.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 6 \\ 3 \cdot 5 + 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}.$$

The identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is of fundamental importance, having the property that

$$\mathbf{I}\vec{\mathbf{v}} = \vec{\mathbf{v}}$$

for any vector $\vec{\mathbf{v}}$.

Identifying

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \vec{\mathbf{x}} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \vec{\alpha} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

then, the system (28) simply becomes

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\alpha}. \tag{29}$$

$\mathbf{A}$ is the coefficient matrix, and $\Delta = ad - bc$ is its determinant, also written $\det(\mathbf{A})$.

The following result is fundamental to the study of systems of linear equations:

Theorem: The system

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\alpha}$$

has a unique solution $\vec{\mathbf{x}}_*$ if and only if $\det(\mathbf{A}) \neq 0$. Otherwise, there are infinite solutions or none. □

For this reason, $\mathbf{A}$ is called singular if $\det(\mathbf{A}) = 0$.

We also define the trace of $\mathbf{A}$ as the sum of the diagonal elements, $\mathrm{tr}(\mathbf{A}) := a + d$

Returning to the differential setting, we see that the homogeneous and autonomous system (26)

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

can be written more succinctly as

$$\vec{\mathbf{x}}\,' = \mathbf{A}\vec{\mathbf{x}}$$

We have seen that $x(t)$ and $y(t)$ each satisfy the second-order ODE with characteristic equation

$$\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0.$$

One generally solves for λ using the quadratic equation: setting $\tau := \mathrm{tr}(\mathbf{A})$ and $\Delta := \det(\mathbf{A})$,

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2},$$

Note that this formula implies $\lambda_1 + \lambda_2 = \tau = \mathrm{tr}(\mathbf{A})$ and $\lambda_1 \cdot \lambda_2 = \Delta = \det(\mathbf{A})$.

We once again have three qualitatively distinct cases:

(1) $\tau^2 - 4\Delta > 0$: real and distinct eigenvalues λ_1, λ_2.

(2) $\tau^2 - 4\Delta = 0$: single real eigenvalue λ.

(3) $\tau^2 - 4\Delta < 0$: complex eigenvalues $\lambda_\pm = \mu \pm i\nu$.

We treat each case and consider its implications for the structure of the 2D system.

Day 31: Systems with real eigenvalues (4.3,4.4.1)

How can we construct solutions to homogeneous and autonomous linear systems?

We continue to consider (26)

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \qquad \Longleftrightarrow \qquad \vec{\mathbf{x}}\,' = \mathbf{A}\vec{\mathbf{x}},$$

according to the structure of the eigenvalues

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2},$$

where $\tau := \mathrm{tr}(\mathbf{A}) = a + d$ and $\Delta := \det(\mathbf{A}) = ad - bc$.

Case (1): Eigenvalues λ_1, λ_2 are real and distinct.

A fundamental set of solutions to the second-order equation

$$x'' - \mathrm{tr}(\mathbf{A})x' + \det(\mathbf{A})x = 0$$

for both x and y is then $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, so we must have

$$\begin{aligned} x(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ y(t) &= C_3 e^{\lambda_1 t} + C_4 e^{\lambda_2 t} \end{aligned} \iff \vec{\mathbf{x}}(t) = e^{\lambda_1 t}\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + e^{\lambda_2 t}\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}$$

This has too many degrees of freedom: an IVP imposes two conditions, but we have four constants.

In fact, C_3 and C_4 depend on C_1 and C_2. To see how, we plug these into (26), or $\vec{\mathbf{x}}\,' = \mathbf{A}\vec{\mathbf{x}}$:

$$\lambda_1 e^{\lambda_1 t}\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + \lambda_2 e^{\lambda_2 t}\begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \vec{\mathbf{x}}\,' = \mathbf{A}\vec{\mathbf{x}} = e^{\lambda_1 t}\mathbf{A}\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + e^{\lambda_2 t}\mathbf{A}\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}$$

Matching the $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ terms, we must have

$$\mathbf{A}\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \lambda_1 \begin{pmatrix} C_1 \\ C_3 \end{pmatrix}, \qquad \mathbf{A}\begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \lambda_2 \begin{pmatrix} C_2 \\ C_4 \end{pmatrix},$$

Each having the form $\mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$:

Definition: Given a matrix $\mathbf{A}$, a pair $\lambda, \vec{\mathbf{v}}$ of a number λ and a non-zero vector $\vec{\mathbf{v}}$ satisfying

$$\mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$$

are respectively called an eigenvalue and eigenvector of $\mathbf{A}$.

□

Example: Show that $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A} = \begin{pmatrix} -2 & 2 \\ 6 & -1 \end{pmatrix}$. What is its eigenvalue?

$$\mathbf{A}\vec{\mathbf{v}} = \begin{pmatrix} -2 & 2 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2+4 \\ 6-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\vec{\mathbf{v}}.$$

Hence the eigenvalue is $\lambda = 2$.

□

Note that the eigenvector condition is equivalent to $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

We have apparently concluded that $\begin{pmatrix} C_1 \\ C_3 \end{pmatrix}$ and $\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$ with eigenvalues λ_1, λ_2.

Example: Find the general solution to the 2D system

$$\begin{aligned} x' &= 2x + y \\ y' &= 2x + 3y, \end{aligned} \qquad \Longleftrightarrow \qquad \vec{\mathbf{x}}' = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \vec{\mathbf{x}},$$

as well as IVP solution with initial conditions $x(0) = 3$, $y(0) = 0$.

The characteristic equation is

$$0 = \lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1),$$

so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 4$. Hence solutions have the form

$$\vec{\mathbf{x}}(t) = e^t \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + e^{4t} \begin{pmatrix} C_2 \\ C_4 \end{pmatrix}.$$

We solve the eigenvector conditions on $\begin{pmatrix} C_1 \\ C_3 \end{pmatrix}$ and $\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}$:

$$(\mathbf{A} - \lambda_1\mathbf{I}) \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \vec{\mathbf{0}} \iff \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff C_1 + C_3 = 0 \implies C_3 = -C_1$$

$$(\mathbf{A} - \lambda_2\mathbf{I}) \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \vec{\mathbf{0}} \iff \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 2C_2 - C_4 = 0 \implies C_4 = 2C_2.$$

Our expression for $\vec{\mathbf{x}}(t)$ now reads

$$\vec{\mathbf{x}}(t) = \begin{pmatrix} C_1 \\ -C_1 \end{pmatrix} e^t + \begin{pmatrix} C_2 \\ 2C_2 \end{pmatrix} e^{4t}$$

$$\implies \boxed{\vec{\mathbf{x}}(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{4t}}$$

This is the general solution to our 2D system. Equivalently,

$$\boxed{\begin{aligned} x(t) &= C_1 e^t + C_2 e^{4t} \\ y(t) &= 2C_2 e^{4t} - C_1 e^t \end{aligned}}$$

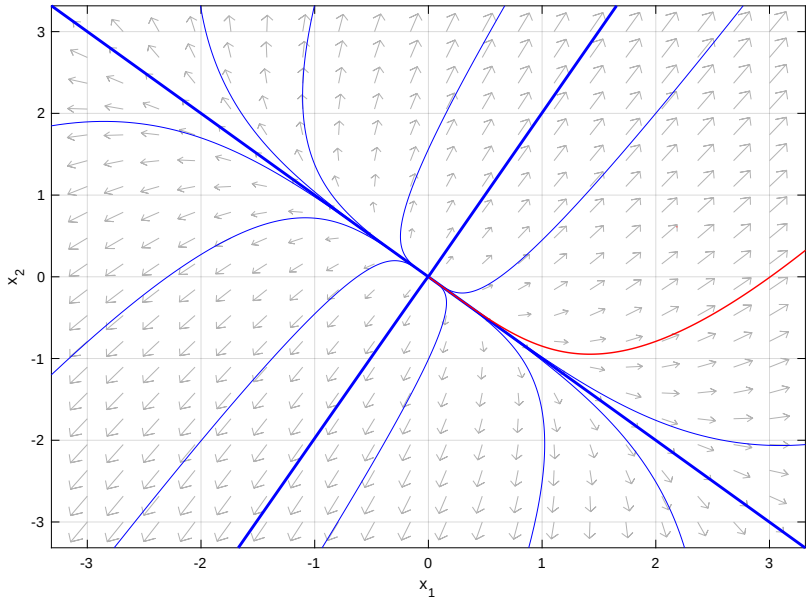
To also solve the given IVP, we may find C_1 and C_2 by imposing the initial conditions:

$$\begin{pmatrix} C_1 + C_2 \\ 2C_2 - C_1 \end{pmatrix} = \vec{\mathbf{x}}(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \implies C_1 = 2,\ C_2 = 1.$$

This finally yields

$$\boxed{\vec{\mathbf{x}}(t) = \begin{pmatrix} 2e^t + e^{4t} \\ 2e^{4t} - 2e^t \end{pmatrix}} \qquad \Longleftrightarrow \qquad \boxed{\begin{aligned} x(t) &= 2e^t + e^{4t} \\ y(t) &= 2e^{4t} - 2e^t \end{aligned}}$$

To visualize this, one may construct a phase portrait:



The particular solution identified by the IVP is shown in red.

The emboldened, straight trajectories point along eigenvectors, corresponding to $C_1 = 0$ or $C_2 = 0$.

Solutions emanate from 0 ($t \to -\infty$) tangent to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ since e^{4t} decays faster than e^t in this limit.

Solutions diverge to infinity ($t \to \infty$) parallel to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ since e^{4t} grows faster than e^t in this limit.

□

Observe that the eigenvalues above were distinct, real, and positive: this always results in a qualitatively similar phase portrait, wherein the critical point $\vec{\mathbf{0}}$ is dubbed a nodal source. It is unstable.

If the eigenvalues are distinct, real, and negative, the picture is qualitatively similar with reversed arrows, yielding a nodal sink. It is asymptotically stable.

Example: Find the general solution to the 2D system

$$\begin{aligned} x' &= -x + 2y \\ y' &= y \end{aligned} \qquad \Longleftrightarrow \qquad \vec{\mathbf{x}}\,' = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \vec{\mathbf{x}},$$

as well as IVP solution with initial conditions $x(0) = 0$, $y(0) = -1$.

Noting $\mathrm{tr}(\mathbf{A}) = 0$ and $\det(\mathbf{A}) = -1$, the characteristic equation is

$$\lambda^2 - 1 = 0,$$

so that $\lambda_1 = -1$, $\lambda_2 = 1$. The eigenvector conditions give

$$(\mathbf{A} - \lambda_1 \mathbf{I}) \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \vec{\mathbf{0}} \iff \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 0C_1 + 3C_3 = 0 \implies C_3 = 0$$

$$(\mathbf{A} - \lambda_2 \mathbf{I}) \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \vec{\mathbf{0}} \iff \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 2C_4 - 2C_2 = 0 \implies C_4 = C_2.$$

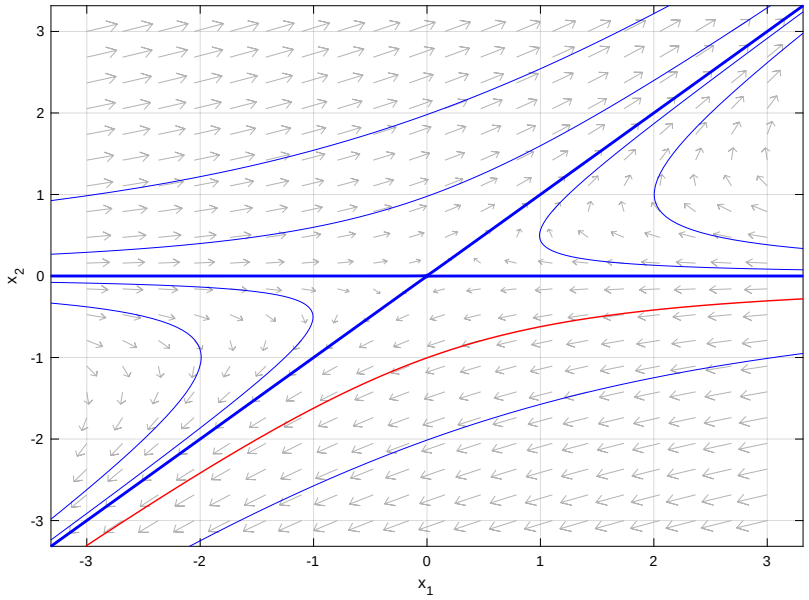
The general solution is then

$$\boxed{\vec{\mathbf{x}}_g(t) = C_1 e^{-t}\begin{pmatrix}1\\0\end{pmatrix} + C_2 e^{t}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}C_1 e^{-t} + C_2 e^{t}\\ C_2 e^{t}\end{pmatrix}}.$$

Setting the initial condition yields $C_1 = 1$, $C_2 = -1$, giving

$$\boxed{\vec{\mathbf{x}}(t) = \begin{pmatrix}e^{-t} - e^{t}\\ -e^{t}\end{pmatrix}} \quad \Longleftrightarrow \quad \boxed{\begin{aligned} x(t) &= e^{-t} - e^{t}\\ y(t) &= -e^{t}\end{aligned}}$$

The phase portrait of this system is:



Solutions come in $(t \to -\infty)$ from infinity tangent to $\begin{pmatrix}1\\0\end{pmatrix}$ since e^t decays to zero in this limit.

Solutions diverge out $(t \to \infty)$ to infinity tangent to $\begin{pmatrix}1\\1\end{pmatrix}$ since e^{-t} decays to zero in this limit.

□

Here the eigenvalues were real with opposite sign; the origin is called a saddle point. It is unstable.

Note: This particular example could also be solved by noting that the equation $y' = y$ only involves y, so can be solved directly. One could then substitute the solution $y(t) = Ce^t$ into $x' = -x + 2y$ and solve for $x(t)$ by variation of parameters.

Day 32: Systems with repeated eigenvalues (4.4.3)

How do we construct and visualize solutions when A has a single eigenvalue?

We continue to consider (26)

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \qquad \Longleftrightarrow \qquad \vec{\mathbf{x}}\,' = \mathbf{A}\vec{\mathbf{x}},$$

according to the structure of the eigenvalues

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2},$$

where $\tau := \mathrm{tr}(\mathbf{A}) = a + d$ and $\Delta := \det(\mathbf{A}) = ad - bc$.

Case (2): Single real, repeated eigenvalue λ.

A fundamental set of solutions to the second-order equation

$$x'' - \mathrm{tr}(\mathbf{A})x' + \det(\mathbf{A})x = 0$$

for both x and y is then $e^{\lambda t}$ and $te^{\lambda t}$, so we must have

$$\begin{aligned} x(t) &= C_1 e^{\lambda t} + C_2 t e^{\lambda t} \\ y(t) &= C_3 e^{\lambda t} + C_4 t e^{\lambda t} \end{aligned} \iff \vec{\mathbf{x}}(t) = e^{\lambda t}\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + te^{\lambda t}\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}$$

This once again has too many degrees of freedom. Plugging into $\vec{\mathbf{x}}\,' = \mathbf{A}\vec{\mathbf{x}}$,

$$e^{\lambda t}\left[\lambda\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + \begin{pmatrix} C_2 \\ C_4 \end{pmatrix}\right] + \lambda t e^{\lambda t}\begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \vec{\mathbf{x}}\,' = \mathbf{A}\vec{\mathbf{x}} = e^{\lambda t}\mathbf{A}\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + te^{\lambda t}\mathbf{A}\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}$$

Matching the $e^{\lambda t}$ and $te^{\lambda t}$ terms, we must have

$$\mathbf{A}\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \lambda\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + \begin{pmatrix} C_2 \\ C_4 \end{pmatrix}, \qquad \mathbf{A}\begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \lambda\begin{pmatrix} C_2 \\ C_4 \end{pmatrix},$$

which after a slight rearrangement reads

$$(\mathbf{A} - \lambda\mathbf{I})\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} C_2 \\ C_4 \end{pmatrix}, \qquad (\mathbf{A} - \lambda\mathbf{I})\begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \vec{\mathbf{0}},$$

The second relation indicates that $\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}$, if nonzero, is an eigenvector with eigenvalue λ.

$\begin{pmatrix} C_1 \\ C_3 \end{pmatrix}$ is called a generalized eigenvector (since the first relation implies $(\mathbf{A} - \lambda\mathbf{I})^2\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \vec{\mathbf{0}}$).

For the ultimate structure of the phase portrait, there are two qualitatively distinct sub-cases:

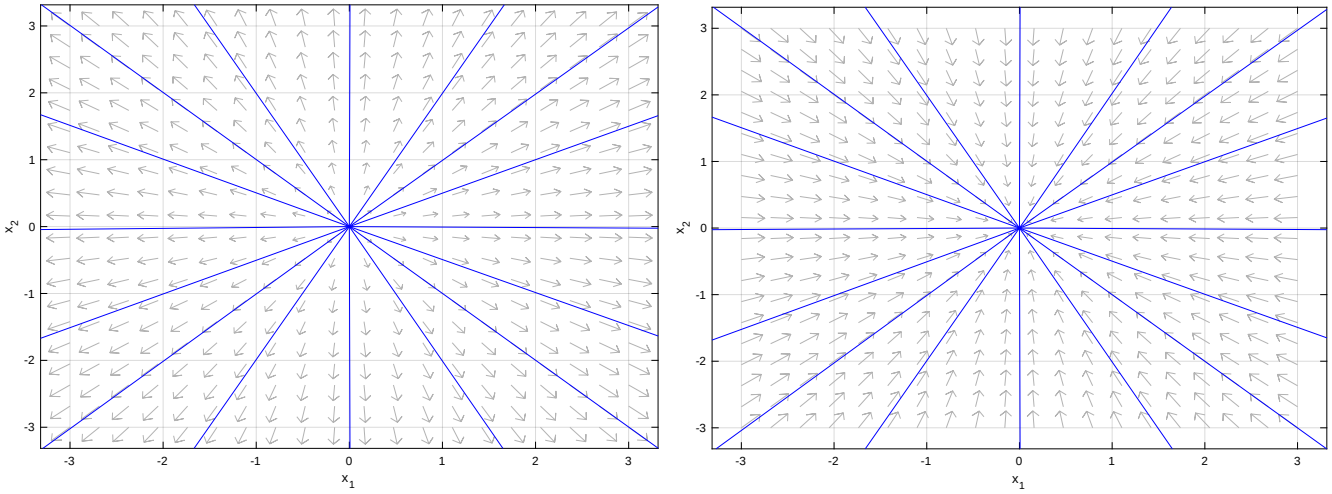
(a) $\mathbf{A} = \lambda\mathbf{I} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

(b) $\mathbf{A} \neq \lambda \mathbf{I}$.

In case (a), the above relations force $C_2 = C_4 = 0$ while imposing no constraint on C_1, C_3, so

$$\begin{aligned} x(t) &= C_1 e^{\lambda t} \\ y(t) &= C_3 e^{\lambda t} \end{aligned} \iff \vec{\mathbf{x}}(t) = e^{\lambda t} \begin{pmatrix} C_1 \\ C_3 \end{pmatrix}$$

The phase portrait is one of:



$\lambda > 0$ on the left, and $\lambda < 0$ on the right. We call the origin a proper node, or star point. It is unstable for $\lambda > 0$, and it is asymptotically stable for $\lambda < 0$.

We treat case (b) through an example.

Example: Find the general solution to the 2D system

$$\begin{aligned} x' &= -x - y \\ y' &= x - 3y \end{aligned} \qquad \iff \qquad \vec{\mathbf{x}}\,' = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} \vec{\mathbf{x}}.$$

Noting $\operatorname{tr}(\mathbf{A}) = -4$ and $\det(\mathbf{A}) = 4$, the characteristic equation is

$$0 = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2,$$

so the unique eigenvalue is $\lambda = -2$. We solve the eigenvector condition for $\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}$ as usual:

$$(\mathbf{A} - \lambda \mathbf{I}) \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \vec{\mathbf{0}} \iff \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff C_2 - C_4 = 0 \iff C_4 = C_2,$$

We similarly solve the generalized eigenvector condition for $\begin{pmatrix} C_1 \\ C_3 \end{pmatrix}$:

$$(\mathbf{A} - \lambda \mathbf{I}) \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} \iff \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} \iff C_1 - C_3 = C_2 \iff C_3 = C_1 - C_2.$$

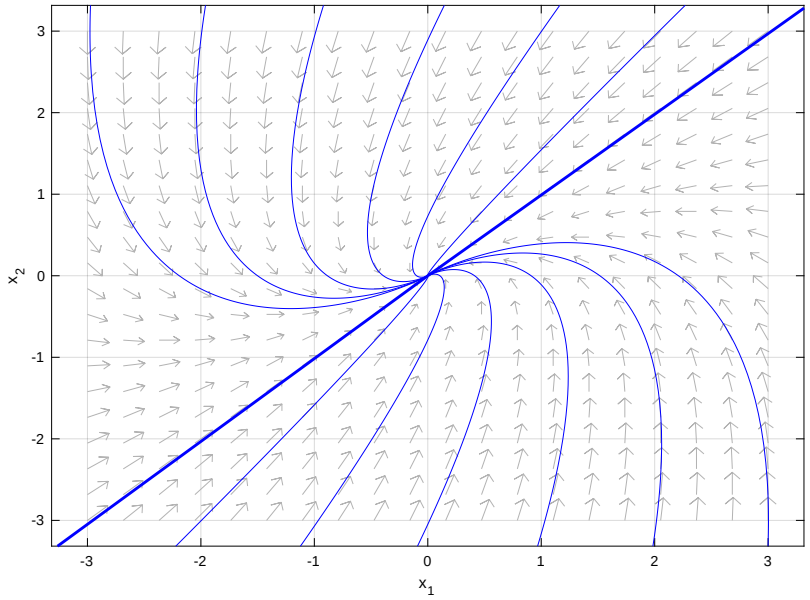
The general solution is therefore

$$\vec{\mathbf{x}}_g(t) = e^{-2t}\begin{pmatrix} C_1 \\ C_1 - C_2 \end{pmatrix} + te^{-2t}\begin{pmatrix} C_2 \\ C_2 \end{pmatrix}$$

$$\boxed{\vec{\mathbf{x}}_g(t) = e^{-2t}\left[(C_1 + C_2 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]} \quad \Longleftrightarrow \quad \boxed{\begin{aligned} x_g(t) &= (C_1 + C_2 t)e^{-2t} \\ y_g(t) &= (C_1 - C_2 + C_2 t)e^{-2t} \end{aligned}}$$

□

The above example's phase portrait is:



The emboldened, straight trajectories are (anti-)parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, corresponding to setting $C_2 = 0$.

All solutions decay to zero as $t \to \infty$ since $\lambda < 0$ in the drawn system.

Solutions are parallel/antiparallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for both $t \to \pm\infty$ due to the dominance of the t terms. Solutions switch direction between these limits since the sign of the t terms changes.

The origin here is called a degenerate node or improper node. When $\lambda < 0$ (as shown), it is asymptotically stable. When $\lambda > 0$, the arrows reverse, and it is unstable.

Day 33: Complex eigenvalues (4.4.2); Recap (4.5)

How do we construct and visualize solutions when A has complex eigenvalues?

We continue to consider (26)

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \qquad \Longleftrightarrow \qquad \vec{\mathbf{x}}\,' = \mathbf{A}\vec{\mathbf{x}},$$

according to the structure of the eigenvalues

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2},$$

where $\tau := \mathrm{tr}(\mathbf{A}) = a + d$ and $\Delta := \det(\mathbf{A}) = ad - bc$.

Case (3): Eigenvalues $\lambda_\pm = \mu \pm i\nu$ are complex conjugates.

A fundamental set of solutions to the second-order equation

$$x'' - \mathrm{tr}(\mathbf{A})x' + \det(\mathbf{A})x = 0$$

for both x and y is then $e^{\mu t}\cos(\nu t)$ and $e^{\mu t}\sin(\nu t)$, so we must have

$$\begin{aligned} x(t) &= e^{\mu t}\left(C_1\cos(\nu t) + C_2\sin(\nu t)\right) \\ y(t) &= e^{\mu t}\left(C_3\cos(\nu t) + C_4\sin(\nu t)\right) \end{aligned} \iff \vec{\mathbf{x}}(t) = e^{\mu t}\left[\cos(\nu t)\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + \sin(\nu t)\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}\right]$$

We omit relating the C_i's in general, but exhibit the treatment of IVP's through an example.[2]

Example Solve the system IVP

$$\begin{aligned} x' &= -x + 2y, \quad x(0) = 1 \\ y' &= -2x - y \quad y(0) = 3 \end{aligned} \qquad \Longleftrightarrow \qquad \vec{\mathbf{x}}\,' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}\vec{\mathbf{x}}, \quad \vec{\mathbf{x}}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Noting $\mathrm{tr}(\mathbf{A}) = -2$ and $\det(\mathbf{A}) = 5$, the characteristic equation is

$$0 = \lambda^2 + 2\lambda + 5 = (\lambda + 1)^2 + 4,$$

so the eigenvalues are $\lambda_\pm = -1 \pm 2i$. Noting that

$$\vec{\mathbf{x}}\,'(0) = \mathbf{A}\vec{\mathbf{x}}(0) = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix},$$

we can now avoid relating the C_i's since $x(t)$ and $y(t)$ are individually characterized by

$$\begin{aligned} x(t) &= e^{-t}\left(C_1\cos(2t) + C_2\sin(2t)\right), \qquad x(0) = 1, \quad x'(0) = \quad 5 \\ y(t) &= e^{-t}\left(C_3\cos(2t) + C_4\sin(2t)\right), \qquad y(0) = 3, \quad y'(0) = \quad -5 \end{aligned}$$

In particular,

$$\begin{aligned} C_1 = x(0) = 1, \quad 2C_2 - C_1 = x'(0) = \quad 5 \quad &\Longrightarrow \quad C_1 = 1, \quad C_2 = \quad 3 \\ C_3 = y(0) = 3, \quad 2C_4 - C_3 = y'(0) = \quad -5 \quad &\Longrightarrow \quad C_3 = 3, \quad C_4 = \quad -1, \end{aligned}$$

[2] The interested student can see the general relationship at the end of this day's notes.

so that the IVP solution is

$$\boxed{\vec{\mathbf{x}}(t) = e^{-t}\begin{pmatrix}\cos(2t)+3\sin(2t)\\ 3\cos(2t)-\sin(2t)\end{pmatrix}} \quad \Longleftrightarrow \quad \boxed{\begin{aligned} x(t) &= e^{-t}(\cos(2t)+3\sin(2t))\\ y(t) &= e^{-t}(3\cos(2t)-\sin(2t))\end{aligned}}$$

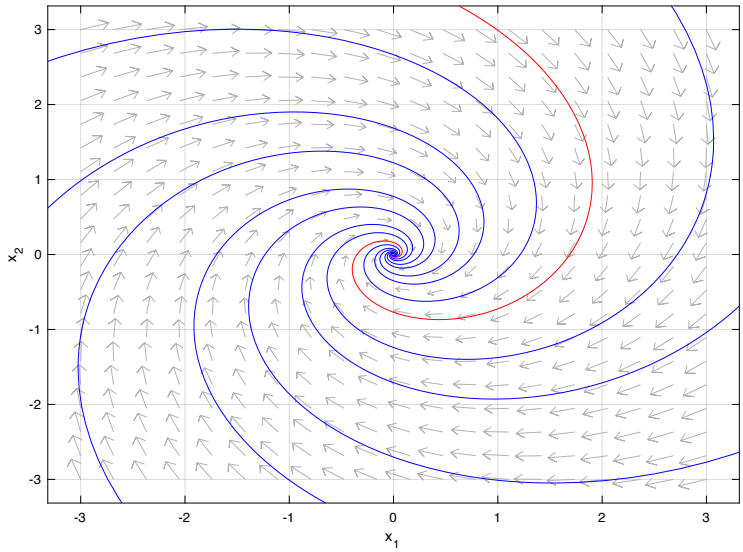
□

Using a phase-amplitude form in each component, the above solution $\vec{\mathbf{x}}(t)$ may be written

$$\vec{\mathbf{x}}(t) = \sqrt{10}e^{-t}\begin{pmatrix}\cos(-2t+\delta)\\ \sin(-2t+\delta)\end{pmatrix},$$

where $\delta \approx 1.249$ satisfies $\cos(\delta) = 1/\sqrt{10}$, $\sin(\delta) = 3/\sqrt{10}$.

The vector part of this $\vec{\mathbf{x}}(t)$ rotates clockwise around the unit circle in the phase plane while the coefficient $\sqrt{10}e^{-t}$ scales the (decaying) radius. Hence, this solution spirals inward. This is generic:



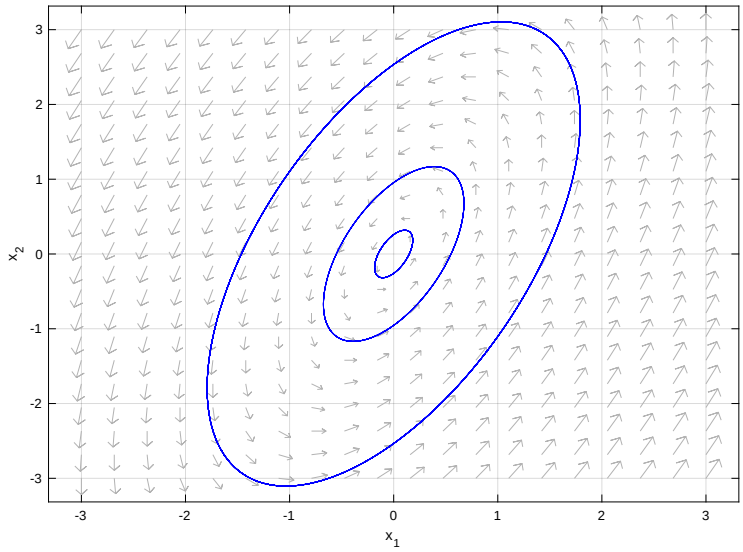
The IVP solution above is shown in red, and a variety of other trajectories are in blue.

In the above, $\mathrm{Re}(\lambda_\pm) = -1$ led to the radial decay proportional to e^{-t}.

The above phase portrait is typical of systems with complex eigenvalues with negative real part. The origin is called a spiral sink. It is asymptotically stable.

The picture is similar with reversed arrows when eigenvalues are complex with positive real part. The origin is then called a spiral source. It is unstable.

If complex eigenvalues have *zero* real part, then there is no radial growth or decay (shown below). The origin is then called a center. It is stable, but *not* asymptotically stable.



This is the phase portrait for the system

$$\vec{\mathbf{x}}' = \begin{pmatrix} 1 & -1 \\ 3 & -1 \end{pmatrix} \vec{\mathbf{x}},$$

which has eigenvalues $\lambda_{\pm} = \pm\sqrt{2}i$. Trajectories are generally ellipses.

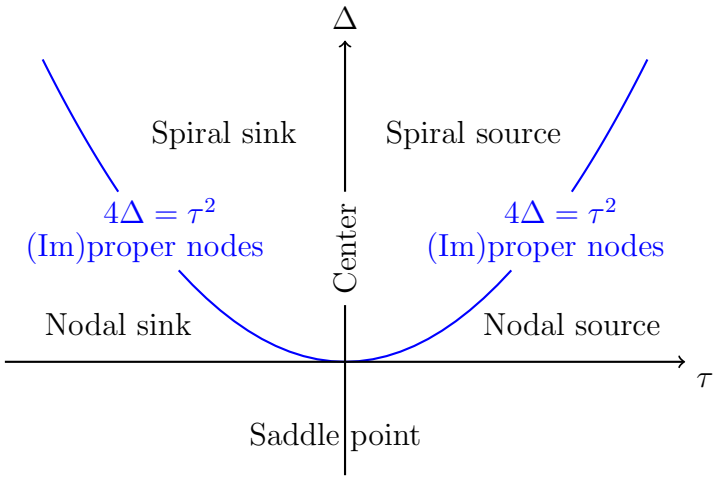
Linear Systems Summary/Recap

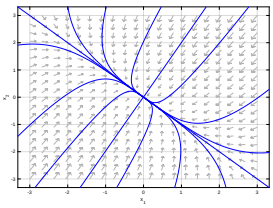
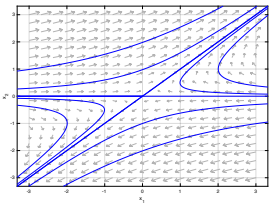
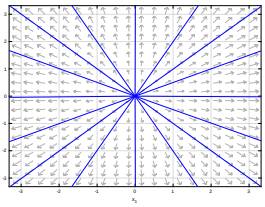
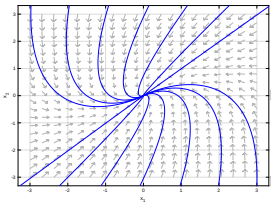
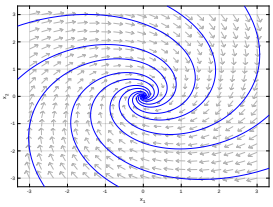
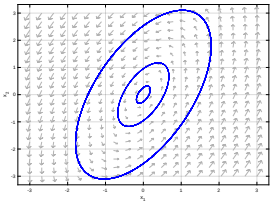
We have now seen all of the possible types of the isolated central critical point in the system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$, entirely characterized by the eigenvalues of $\mathbf{A}$: see the summary table on the next page.

Equivalently, these may be characterized directly by $\tau = \text{tr}(\mathbf{A})$ and $\Delta = \det(\mathbf{A})$ since

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}.$$

It is worthwhile to observe from this expression that $\tau = \lambda_1 + \lambda_2$ and $\Delta = \lambda_1 \cdot \lambda_2$.



Eigenvalues	Critical Point Type	Stability	Phase Portrait
$\lambda_1 \neq \lambda_2 < 0$	Node (sink)	Asymp. Stable	
$\lambda_1 \neq \lambda_2 > 0$	Node (source)	Unstable	Reverse arrows above
$\lambda_1 < 0 < \lambda_2$	Saddle point	Unstable	
$\lambda_1 = \lambda_2 < 0; \quad \mathbf{A} = \lambda\mathbf{I}$	Proper node	Asymp. Stable	Reverse arrows below
$\lambda_1 = \lambda_2 > 0; \quad \mathbf{A} = \lambda\mathbf{I}$	Proper node	Unstable	
$\lambda_1 = \lambda_2 < 0; \quad \mathbf{A} \neq \lambda\mathbf{I}$	Improper node	Asymp. Stable	
$\lambda_1 = \lambda_2 > 0; \quad \mathbf{A} \neq \lambda\mathbf{I}$	Improper node	Unstable	Reverse arrows above
$\lambda_\pm$ complex; $\mathrm{Re}(\lambda_\pm) < 0$	Spiral point (sink)	Asymp. Stable	
$\lambda_\pm$ complex; $\mathrm{Re}(\lambda_\pm) > 0$	Spiral point (source)	Unstable	Reverse arrows above
$\lambda_\pm$ complex; $\mathrm{Re}(\lambda_\pm) = 0$	Center	Stable	

Optional Material: relating coefficients in case (3) (not discussed in lecture):

Above, we saw that solutions $x(t)$ and $y(t)$ in case (3) must have the form

$$\begin{array}{l} x(t) = e^{\mu t}\left(C_1\cos(\nu t) + C_2\sin(\nu t)\right) \\ y(t) = e^{\mu t}\left(C_3\cos(\nu t) + C_4\sin(\nu t)\right) \end{array} \iff \vec{\mathbf{x}}(t) = e^{\mu t}\left[\cos(\nu t)\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + \sin(\nu t)\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}\right]$$

One can obtain relations between the C_i's by plugging into $\vec{\mathbf{x}}\,' = \mathbf{A}\vec{\mathbf{x}}$ as before:

$$\begin{aligned} e^{\mu t}\cos(\nu t)\left[\mu\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + \nu\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}\right] + e^{\mu t}\sin(\nu t)\left[\mu\begin{pmatrix} C_2 \\ C_4 \end{pmatrix} - \nu\begin{pmatrix} C_1 \\ C_3 \end{pmatrix}\right] \\ = e^{\mu t}\left[\cos(\nu t)\mathbf{A}\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + \sin(\nu t)\mathbf{A}\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}\right], \end{aligned}$$

and matching $e^{\mu t}\cos(\nu t)$ and $e^{\mu t}\sin(\nu t)$ terms yields

$$(\mathbf{A} - \mu\mathbf{I})\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \nu\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}, \qquad (\mathbf{A} - \mu\mathbf{I})\begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = -\nu\begin{pmatrix} C_1 \\ C_3 \end{pmatrix}.$$

These are equivalent (challenge: why?): either can be used to reduce from four to two constants.

Example Find the general solution for the 2D system

$$\begin{array}{rl} x' = & -x + 2y \\ y' = & -2x - y \end{array} \qquad \iff \qquad \vec{\mathbf{x}}\,' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}\vec{\mathbf{x}}.$$

Noting $\mathrm{tr}(\mathbf{A}) = -2$ and $\det(\mathbf{A}) = 5$, the characteristic equation is

$$0 = \lambda^2 + 2\lambda + 5 = (\lambda + 1)^2 + 4,$$

so the eigenvalues are $\lambda_\pm = -1 \pm 2i$. Solutions therefore take the form

$$\vec{\mathbf{x}}(t) = e^{-t}\left[\cos(2t)\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} + \sin(2t)\begin{pmatrix} C_2 \\ C_4 \end{pmatrix}\right],$$

with the coefficients related according to

$$\nu\begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = (\mathbf{A} - \mu\mathbf{I})\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} 2C_3 \\ -2C_1 \end{pmatrix},$$

giving $C_3 = C_2$ and $C_4 = -C_1$. Hence the general solution is given by

$$\vec{\mathbf{x}}_g(t) = e^{-t}\left[\cos(2t)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \sin(2t)\begin{pmatrix} C_2 \\ -C_1 \end{pmatrix}\right]$$

$$\boxed{\vec{\mathbf{x}}_g(t) = e^{-t}\begin{pmatrix} C_1\cos(2t) + C_2\sin(2t) \\ C_2\cos(2t) - C_1\sin(2t) \end{pmatrix}} \iff \boxed{\begin{array}{ll} x_g(t) = & e^{-t}(C_1\cos(2t) + C_2\sin(2t)) \\ y_g(t) = & e^{-t}(C_2\cos(2t) - C_1\sin(2t)) \end{array}}$$

Note that the above can be expressed as

$$\vec{\mathbf{x}}_g(t) = Ce^{-t}\begin{pmatrix} \cos(-2t + \delta) \\ \sin(-2t + \delta) \end{pmatrix},$$

where $C = \sqrt{C_1^2 + C_2^2}$ and δ satisfies $\cos(\delta) = C_1/C$, $\sin(\delta) = C_2/C$.

This justifies the full phase portrait shown in the IVP version of this problem above.

□

Day 34: Introduction to nonlinear systems and linearization (5.1)

Recall that a general two-dimensional system of first-order ODE's has the form

$$\begin{aligned} x' &= f_1(t,x,y) \\ y' &= f_2(t,x,y) \end{aligned} \iff \vec{\mathbf{x}}' = \vec{\mathbf{f}}(t,\vec{\mathbf{x}}).$$

We called this system linear if it had the form (22),

$$\vec{\mathbf{x}}' = \mathbf{P}(t)\vec{\mathbf{x}} + \vec{\mathbf{g}}(t). \qquad \text{(22 revisited)}$$

Otherwise, it is nonlinear. We stated a linear existence and uniqueness theorem on Day 28. In general, we have (compare to the 1D version on Day 11):

Theorem: Suppose that each f_i and $\dfrac{\partial f_i}{\partial x_j}$ $(i,j=1,2)$ is continuous in some 3D rectangular prism $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, $\alpha_2 < x_2 < \beta_2$ containing $(t_0, \vec{\mathbf{x}}_0)$. There is a unique solution to the IVP

$$\vec{\mathbf{x}}' = \vec{\mathbf{f}}(t,\vec{\mathbf{x}}), \qquad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$$

defined on some interval $(t_0 - h, t_0 + h)$ contained in (α, β).

□

There are precious few methods for constructing these solutions at this level of generality. Qualitative techniques and numerical approximations are essential to their study.

Autonomous systems,

$$\begin{aligned} x' &= f_1(x,y) \\ y' &= f_2(x,y), \end{aligned} \qquad \iff \qquad \vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) \tag{30}$$

are most amenable to qualitative understanding, e.g. through phase plane structure.

An autonomous system's critical points, vectors $\vec{\mathbf{x}}_*$ satisfying $\vec{\mathbf{f}}(\vec{\mathbf{x}}_*) = \vec{\mathbf{0}}$, are of essential importance.

Example: Find the critical points of the autonomous system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y(y-1)(y^2-4) \\ x(y-x^2) \end{pmatrix}$$

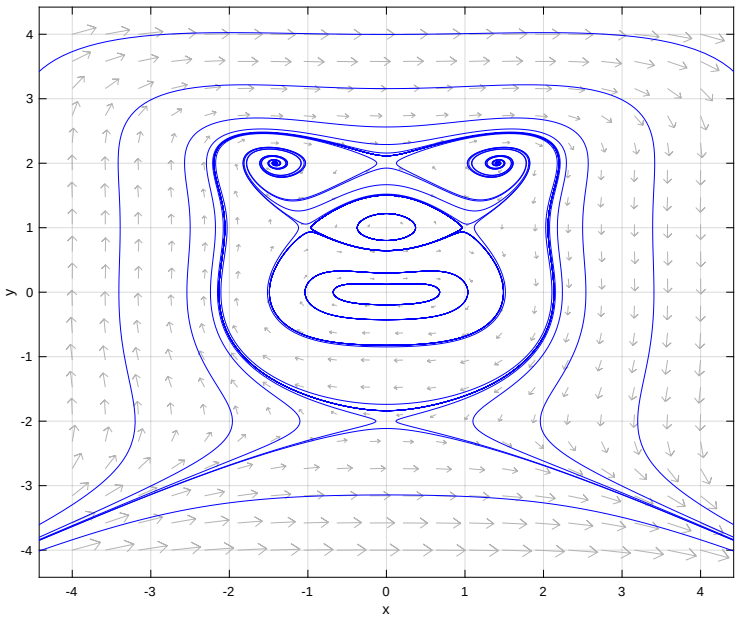
Setting $x' = 0$, we find critical points must satisfy $y = 0$, $y = 1$, or $y = \pm 2$. We consider each case:

- If $y = 0$, setting $y' = 0$ reads $0 = -x^3$, so $(0,0)$ is a critical point.
- If $y = 1$, we have $0 = x(1-x^2)$, so $(0,1)$, $(1,1)$, and $(-1,1)$ are critical points.
- If $y = 2$, we have $0 = x(2-x^2)$, so $(0,2)$, $(\sqrt{2},2)$, and $(-\sqrt{2},2)$ are critical points.
- If $y = -2$, we have $0 = -x(2+x^2)$, so $(0,-2)$ is a critical point.

Hence, the critical points are $\vec{\mathbf{x}}_* = (0,-2)$, $(0,0)$, $(0,1)$, $(\pm 1,1)$, $(0,2)$, and $(\pm\sqrt{2},2)$.

□

Can we visualize this? Finding $\vec{\mathbf{x}}(t)$ may be impossible, but computers can generate phase portraits:



While this does not look like any one of the linear phase portraits we know as a whole, the behavior around each critical point does. In particular, inspection yields the apparent identifications:

- $(\pm\sqrt{2}, 2)$ are spirals. $(-\sqrt{2}, 2)$ is asymptotically stable, while $(\sqrt{2}, 2)$ is unstable.
- $(0, 0)$ and $(0, 1)$ are both centers, and hence stable.
- $(0, -2)$, $(1, 1)$, $(-1, 1)$ and $(0, 2)$ are all saddle points, and hence unstable.

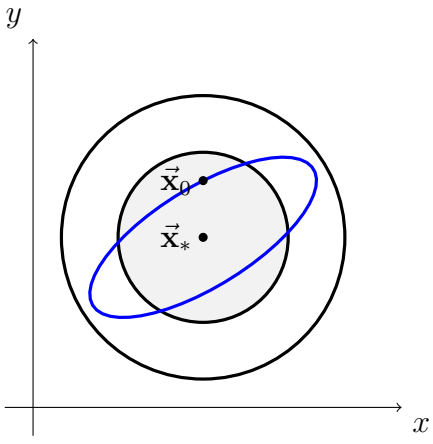
In what sense are these identifications meaningful? Can we make them analytically?

Observe that while the phase man may be unsettling, he cannot hurt you.

We wish to make precise the above qualitative identifications of the nature of critical points $\vec{\mathbf{x}}_*$.

To begin, we require $\vec{\mathbf{x}}_*$ to be isolated: there are no other critical points if one zooms in close enough.

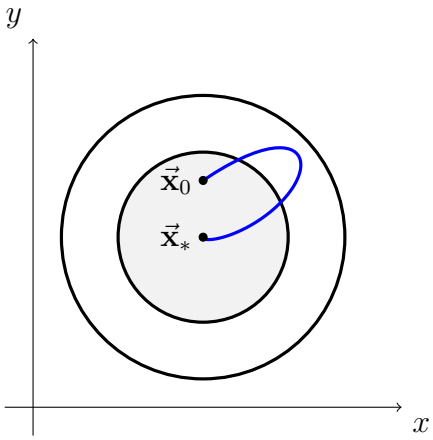
The critical point $\vec{\mathbf{x}}_*$ is stable if trajectories initialized near $\vec{\mathbf{x}}_*$ stay near $\vec{\mathbf{x}}_*$: if there is a region around $\vec{\mathbf{x}}_*$ satisfying that if $\vec{\mathbf{x}}_0$ begins there, the distance from $\vec{\mathbf{x}}(t)$ to $\vec{\mathbf{x}}_*$ remains uniformly bounded[3].



[3] **And** if this uniform bound can be made arbitrarily small by taking the region small enough.

Otherwise, $\vec{\mathbf{x}}_*$ is said to be unstable.

A stable critical point $\vec{\mathbf{x}}_*$ is further said to be asymptotically stable if trajectories initialized near $\vec{\mathbf{x}}_*$ limit to $\vec{\mathbf{x}}_*$: there is also a region around $\vec{\mathbf{x}}_*$ satisfying that if $\vec{\mathbf{x}}_0$ begins there, then $\lim_{t\to\infty} \vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_*$.



How do such properties (and phase portrait features) carry over from linear to nonlinear systems?

To answer this, we must linearize the system (30) around a critical point $\vec{\mathbf{x}}_*$.

Recall: a single-variable function $f(x)$ may be approximated at a given point x_* by its tangent line,

$$f(x) \approx f(x_*) + f'(x_*)(x - x_*).$$

This is the linear function closest to f near x_*. Similarly for the multivariable functions $f_i(x, y)$,

$$f_1(x,y) \approx f_1(x_*,y_*) + \frac{\partial f_1}{\partial x}(x_*,y_*)(x-x_*) + \frac{\partial f_1}{\partial y}(x_*,y_*)(y-y_*)$$
$$f_2(x,y) \approx f_2(x_*,y_*) + \frac{\partial f_2}{\partial x}(x_*,y_*)(x-x_*) + \frac{\partial f_2}{\partial y}(x_*,y_*)(y-y_*)$$

These approximations are the 2D linear functions closest to f_1, f_2 near $\vec{\mathbf{x}}_*$. Setting

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \vec{\mathbf{u}}(t) := \vec{\mathbf{x}}(t) - \vec{\mathbf{x}}_* = \begin{pmatrix} x(t) - x_* \\ y(t) - y_* \end{pmatrix}$$

and taking $\vec{\mathbf{x}}_*$ to be a critical point (so that $\vec{\mathbf{f}}(\vec{\mathbf{x}}_*) = \vec{\mathbf{0}}$), these become

$$\begin{aligned} f_1(x,y) &\approx \tfrac{\partial f_1}{\partial x}(\vec{\mathbf{x}}_*)u + \tfrac{\partial f_1}{\partial y}(\vec{\mathbf{x}}_*)v \\ f_2(x,y) &\approx \tfrac{\partial f_2}{\partial x}(\vec{\mathbf{x}}_*)u + \tfrac{\partial f_2}{\partial y}(\vec{\mathbf{x}}_*)v \end{aligned} \qquad \Longleftrightarrow \qquad \vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx \begin{pmatrix} \frac{\partial f_1}{\partial x}(\vec{\mathbf{x}}_*) & \frac{\partial f_1}{\partial y}(\vec{\mathbf{x}}_*) \\ \frac{\partial f_2}{\partial x}(\vec{\mathbf{x}}_*) & \frac{\partial f_2}{\partial y}(\vec{\mathbf{x}}_*) \end{pmatrix} \vec{\mathbf{u}}$$

Defining the Jacobian matrix for $\vec{\mathbf{f}}$ at $\vec{\mathbf{x}}_*$ to be

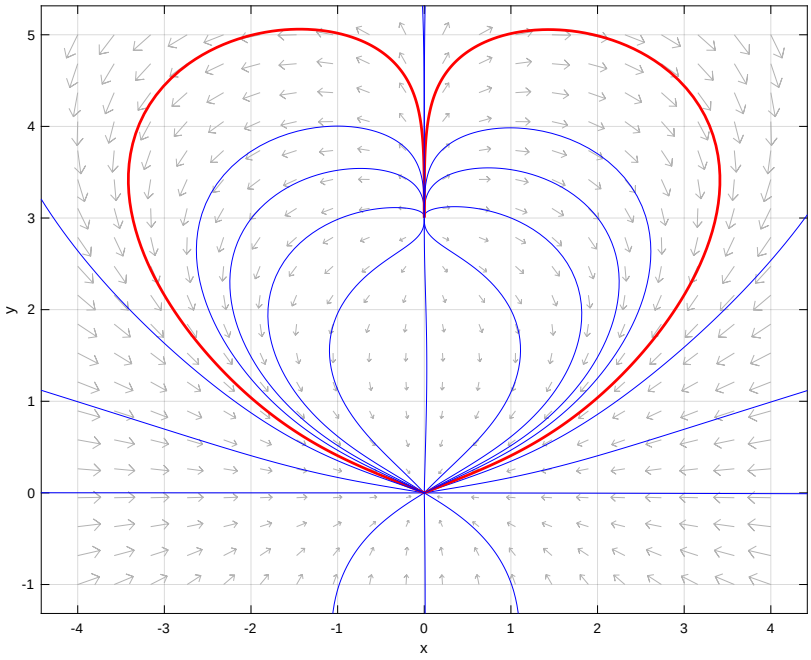
$$\mathbf{J}(\vec{\mathbf{x}}_*) := \begin{pmatrix} \frac{\partial f_1}{\partial x}(\vec{\mathbf{x}}_*) & \frac{\partial f_1}{\partial y}(\vec{\mathbf{x}}_*) \\ \frac{\partial f_2}{\partial x}(\vec{\mathbf{x}}_*) & \frac{\partial f_2}{\partial y}(\vec{\mathbf{x}}_*) \end{pmatrix},$$

we have found that $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx \mathbf{J}(\vec{\mathbf{x}}_*)\vec{\mathbf{u}}$.

Noting that $\vec{\mathbf{u}}\,' = (\vec{\mathbf{x}} - \vec{\mathbf{x}}_*)' = \vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx \mathbf{J}(\vec{\mathbf{x}}_*)\vec{\mathbf{u}}$, we call the system

$$\boxed{\vec{\mathbf{u}}\,' = \mathbf{J}(\vec{\mathbf{x}}_*)\vec{\mathbf{u}}}$$

the linearization of (30) near the critical point $\vec{\mathbf{x}}_*$. We will assume that $\det(\mathbf{J}) \neq 0$.



This is a Valentine's phase portrait with a star point sink at $(0, 0)$ and an unstable node at $(0, 3)$.

For the curious, this was generated from the system:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x(x^2 - y^2) - 3xe^{-(x^2+y^2)/4} \\ (y - 3 - x^2)y \end{pmatrix}$$

Why did I include the ugly $-3xe^{-(x^2+y^2)/4}$ term...?

Day 35: Linearization (5.1,5.2)

What properties carry over from the linearized system to the full system?

Example: The angle θ with the vertical made by an undamped pendulum of length L satisfies

$$\theta'' + \frac{g}{L}\sin(\theta) = 0.$$

Setting $\omega^2 := \frac{g}{L}$ and taking $\begin{pmatrix} x \\ y \end{pmatrix} = \vec{\mathbf{x}} := \begin{pmatrix} \theta \\ \theta' \end{pmatrix}$, this is described by the nonlinear system

$$\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) = \begin{pmatrix} y \\ -\omega^2 \sin(x) \end{pmatrix}.$$

Critical points satisfy $y = 0$ and $\sin(x) = 0$, giving $\vec{\mathbf{x}}_* = (n\pi, 0)$ for any integer n. Note that $\theta = 2n\pi$ corresponds to the nadir of the pendulum, while $\theta = (2n+1)\pi$ corresponds to its zenith.

The Jacobian matrix is given by

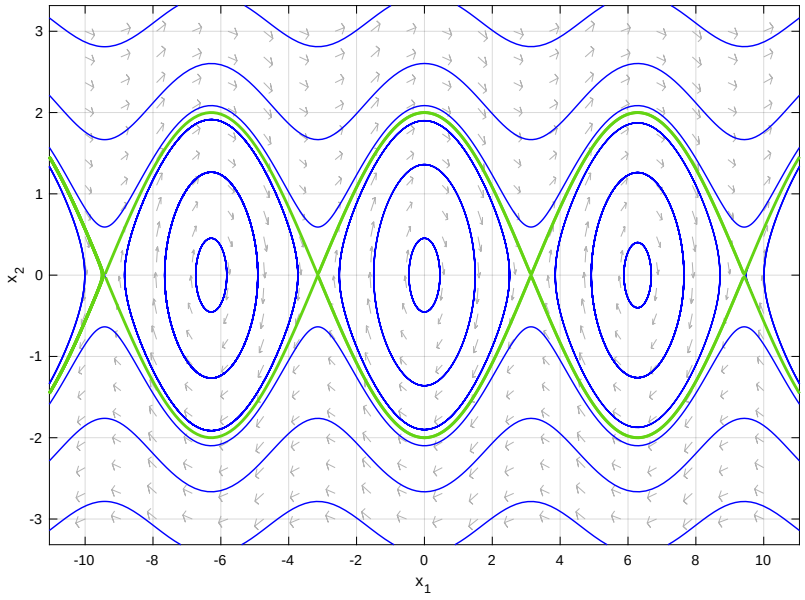
$$\mathbf{J}(\vec{\mathbf{x}}_*) = \begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ -\omega^2 \frac{\partial \sin(x)}{\partial x} & -\omega^2 \frac{\partial \sin(x)}{\partial y} \end{pmatrix}\Bigg|_{\vec{\mathbf{x}}_*} = \begin{pmatrix} 0 & 1 \\ -\omega^2\cos(n\pi) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1}\omega^2 & 0 \end{pmatrix},$$

so that the linearized system is

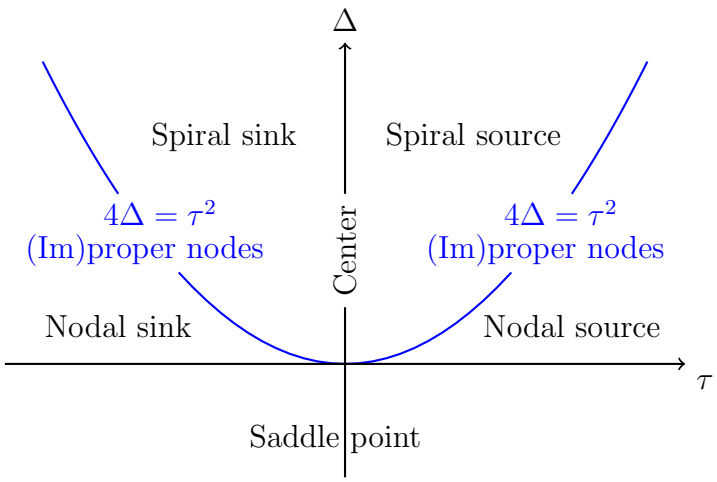
$$\vec{\mathbf{u}}\,' = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1}\omega^2 & 0 \end{pmatrix} \vec{\mathbf{u}}.$$

The linearized system is a center (hence stable) for even n and a saddle (hence unstable) for odd n.

This aligns with the phase portrait. As expected physically, the nadir is stable and zenith unstable.



□



Recall the τ-Δ plane classification of linear critical points, where $\tau = \text{tr}(\mathbf{A})$ and $\Delta = \det(\mathbf{A})$.

This classification carries over from the linearization to the nonlinear system provided that one sits in the **interior** of one of the 2D regions: a saddle point, nodal source/sink, or spiral source/sink.

As before, sinks are asymptotically stable while saddles and sources are unstable.

In edge cases ((im)proper nodes, centers), linearization isn't enough to conclude nonlinear structure.

Linearized centers may be perturbed to spirals in the nonlinear system; stability is inconclusive.

(Im)proper nodes may be perturbed to either nodes or spirals, but stability is preserved.

Example: Evaluate the stability of the critical point $\vec{\mathbf{x}}_* = 0$ in the system

$$\begin{aligned} x' &= -y - x^3 \\ y' &= x - y^3 \end{aligned}$$

One can check at $\vec{\mathbf{x}}_* = \vec{\mathbf{0}}$ is the only critical point. The Jacobian matrix there is

$$\mathbf{J}(\vec{\mathbf{0}}) = \begin{pmatrix} -3x^2 & -1 \\ 1 & -3y^2 \end{pmatrix}\Bigg|_{x=y=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

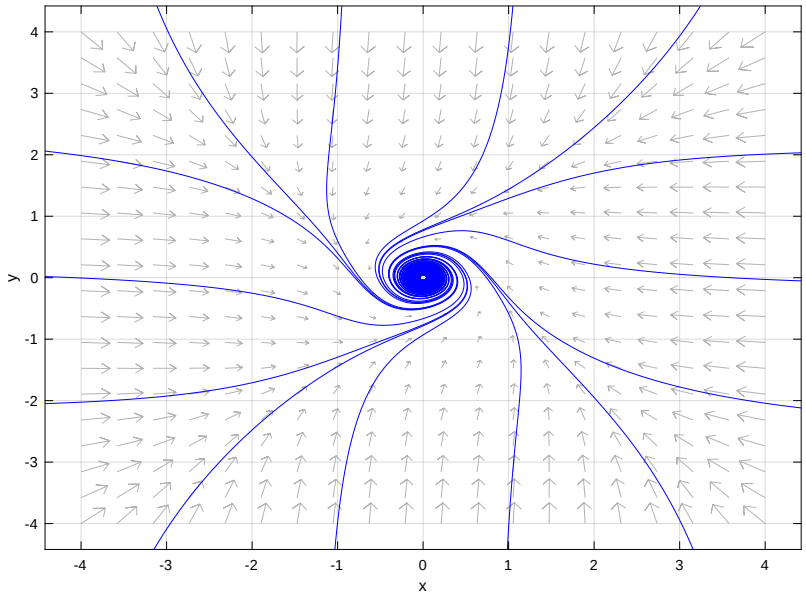
which has trace 0 and determinant 1, yielding a center in the linearized system.

Note that the function $R(x, y) = x^2 + y^2$ satisfies

$$\frac{d}{dt}\left[R(x,y)\right] = 2xx' + 2yy' = 2x(-y - x^3) + 2y(x - y^3) = -2(x^4 + y^4) \leq -R^2 < 0,$$

so R is strictly decreasing along trajectories, limiting to 0. Hence $\vec{\mathbf{x}}_* = \vec{\mathbf{0}}$ is asymptotically stable.

Trajectories in the phase plane spiral into the origin rather than simply encircle it.



Note that the same analysis applied to the system

$$x' = -y + x^3$$
$$y' = x + y^3$$

would conclude that a linearized center can also yield the structure of an unstable spiral.

$R(x, y)$ is called a Lyapunov function: it is monotone along trajectories and extremized at $\vec{\mathbf{x}}_*$.

□

Example: In the pendulum example above, we apparently could not have known that the nonlinear system would have a stable center at $\theta = 2n\pi$ based purely on the linearization.

We could deduce this by considering that phase plane trajectories are characterized by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\omega^2 \sin(x)}{y},$$

and separating variables yields that x and y are related by

$$\frac{1}{2}y^2 = \int y dy = -\omega^2 \int \sin(x) dx = \omega^2 \cos(x) + C$$
$$\implies \qquad E(x, y) := \frac{1}{2}y^2 - \omega^2 \cos(x) = C.$$

For the physically minded, this is precisely the statement of conservation of energy.

The key observation is that the points $\vec{\mathbf{x}}_* = (2n\pi, 0)$ are strict local minima of $E(x, y)$.

Near each $\vec{\mathbf{x}}_*$, then, the plot of the surface $z = E(x, y)$ resembles a bottoming out bowl. Low enough level sets, "cuts" through the bowl at a given height, form closed loops: see this Geogebra applet.

□

Day 36: Modelling with nonlinear systems (5.2,5.3)

What can we model with nonlinear systems?

Recall that we long ago modeled the variation of a single population via (3), the logistic equation

$$x' = r\left(1 - \frac{x}{K}\right)x, , \qquad \text{((3) revisited)}$$

where K is the population's carrying capacity and $r\left(1 - \frac{x}{K}\right)$ is its effective growth rate.

We now consider two populations $x(t)$, $y(t)$ competing for resources. In isolation, each is governed by (3). With both present, each detracts from the other's growth. This is captured by

$$\begin{aligned} \frac{dx}{dt} &= (r_1 - \alpha_1 x - \beta_1 y)x \\ \frac{dy}{dt} &= (r_2 - \alpha_2 y - \beta_2 x)y, \end{aligned} \tag{31}$$

where r_i, α_i, and β_i are positive constants. Here, r_i/α_i plays the role of each species' capacity K.

We might wonder whether this model can lead to coexistence, or if one species necessarily "wins".

Such questions can be answered by understanding the structure of the system (31)'s critical points.

Example: Determine the initial conditions leading to coexistence of two species modeled by

$$\begin{aligned} \frac{dx}{dt} &= (21 - 3x - 4y)x \\ \frac{dy}{dt} &= (12 - 3y - x)y, \end{aligned}$$

We determine the system's critical points by requiring

$$\begin{aligned} 0 &= (21 - 3x - 4y)x \\ 0 &= (12 - 3y - x)y, \end{aligned}$$

yielding $\vec{\mathbf{x}}_* = (0,0)$, $(7,0)$, $(0,4)$, and $(3,3)$. The Jacobian matrix is given by

$$\mathbf{J}(\vec{\mathbf{x}}_*) = \begin{pmatrix} 21 - 6x - 4y & -4x \\ -y & 12 - 6y - x \end{pmatrix}\Bigg|_{\vec{\mathbf{x}}_*}$$

Around each critical point, then, the matrix defining its linearization is

$$\mathbf{J}((0,0)) = \begin{pmatrix} 21 & 0 \\ 0 & 12 \end{pmatrix} \qquad \mathbf{J}((3,3)) = \begin{pmatrix} -9 & -12 \\ -3 & -9 \end{pmatrix}$$
$$\mathbf{J}((7,0)) = \begin{pmatrix} -21 & -28 \\ 0 & 5 \end{pmatrix} \qquad \mathbf{J}((0,4)) = \begin{pmatrix} 5 & 0 \\ -4 & -12 \end{pmatrix}$$

$\mathbf{J}((0,0))$ has $\lambda_1 = 21$ and $\lambda_2 = 12$, giving an unstable node, with $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\mathbf{J}((3,3))$ has $\lambda_1 = -3$ and $\lambda_2 = -15$, giving a stable node, with $\vec{\mathbf{v}}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$\mathbf{J}((7,0))$ has $\lambda_1 = -21$ and $\lambda_2 = 5$, giving a saddle, with $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} -14 \\ 13 \end{pmatrix}$.

$\mathbf{J}((0,4))$ has $\lambda_1 = 5$ and $\lambda_2 = -12$, giving a saddle, with $\vec{\mathbf{v}}_1 = \begin{pmatrix} 17 \\ -4 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

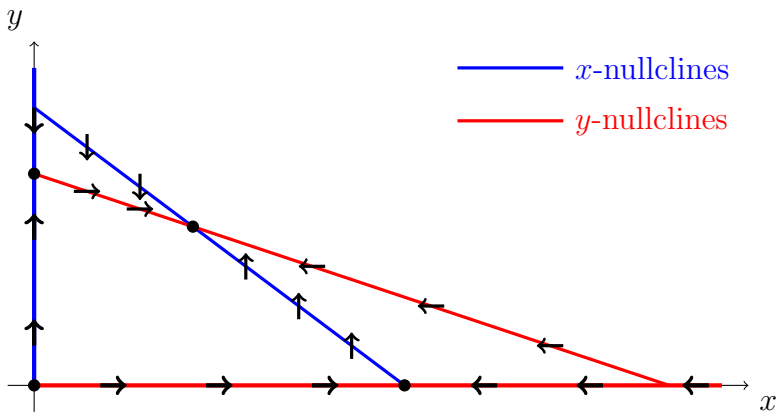
Collating this into a phase portrait is aided by considering nullclines, curves in the x-y plane along which either $x' = 0$ (x-nullclines) or $y' = 0$ (y-nullclines). The x-nullclines (in blue below) solve

$$0 = (21 - 3x - 4y)x,$$

giving the y-axis together with a line of slope $-3/4$. The y-nullclines (in red below) solve

$$0 = (12 - 3y - x)y,$$

the x-axis together with a line of slope $-1/3$.



This plot shows nullclines together with the direction of trajectories through them: vertical on x-nullclines (blue) and horizontal on y-nullclines (red).

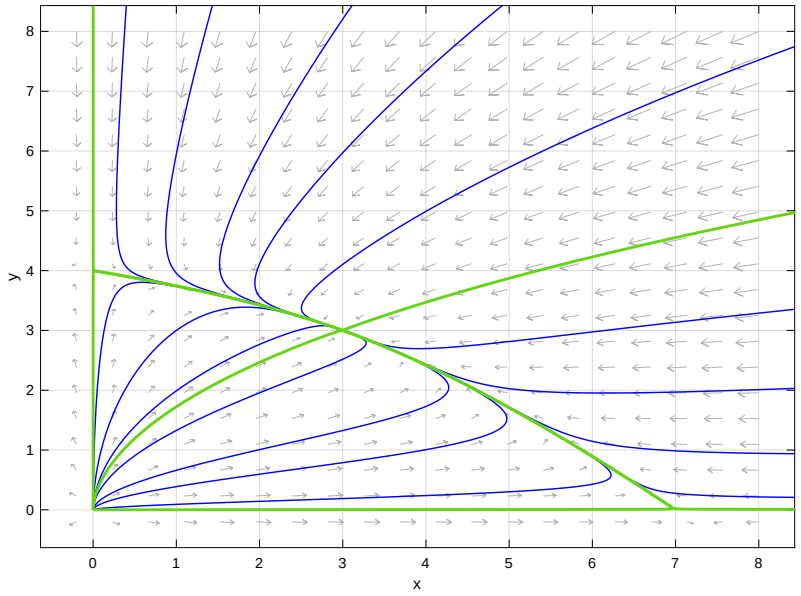
First note that the x- and y-axes are trajectories of the system: if a species is not present initially, it never spontaneously emerges. This ensures that solutions with $x_0, y_0 > 0$ keep $x(t), y(t) > 0$.

Critical points of the system correspond to intersections between oppositely colored lines.

Below the sloped blue line, the horizontal motion is to the right; above, it is to the left.

Below the sloped red line, the vertical motion is upward; above, it is downward.

Combining this with the critical point structure, we can now sketch a fairly accurate phase portrait.



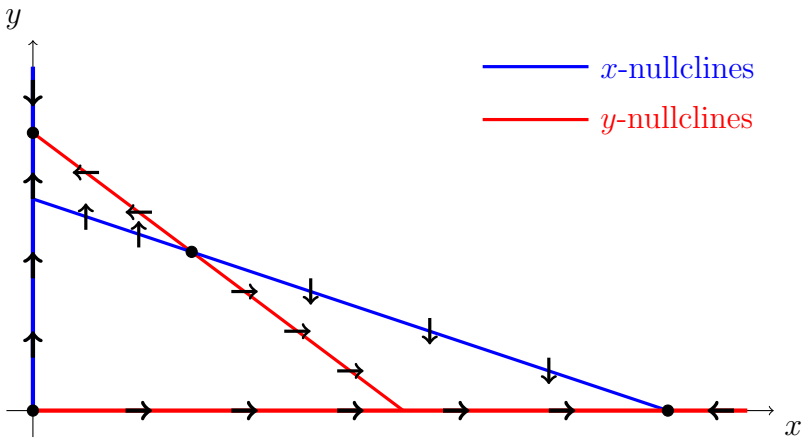
This indicates that *all* positive initial conditions lead to coexistence at the equilibrium $(3, 3)$.

□

Varying coefficients in the general model (31) amounts to shifting and rotating the sloped nullclines.

One can deduce the qualitative behavior of trajectories in these models from nullclines alone, based on whether the red line is steeper than the blue or vice-versa, or whether they intersect or not.

E.g., consider how long-term behavior would be different under the following nullcline structure:



Day 37: Modelling with nonlinear systems (cont.'d)

What can we model with nonlinear systems?

Another type of population dynamic is modelled by the Lotka-Volterra, or predator-prey, equations:

$$\begin{aligned}\frac{dx}{dt} &= (a-\alpha y)x \\ \frac{dy}{dt} &= (-c+\gamma x)y,\end{aligned} \tag{32}$$

where a, α, c, and γ are positive constants.

These equations model the interaction of a predator population $y(t)$, which decays at rate $-c$ in isolation, with a prey population $x(t)$, which grows at rate a in isolation.

Prey presence increases the predator growth rate, while predators decrease the prey growth rate.

Example: Determine the long-term behavior of predator and prey populations modeled by

$$\begin{aligned}\frac{dx}{dt} &= (6-3y)x \\ \frac{dy}{dt} &= (2x-6)y,\end{aligned}$$

We determine the system's critical points by requiring

$$\begin{aligned}0 &= (6-3y)x \\ 0 &= (2x-6)y,\end{aligned}$$

yielding $\vec{\mathbf{x}}_* = (0,0), (3,2)$. The Jacobian matrix is given by

$$\mathbf{J}(\vec{\mathbf{x}}_*) = \begin{pmatrix} 6-3y & -3x \\ 2y & 2x-6 \end{pmatrix}\Bigg|_{\vec{\mathbf{x}}_*}$$

Around each critical point, then, the matrix defining its linearization is

$$\mathbf{J}((0,0)) = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix} \qquad \mathbf{J}((3,2)) = \begin{pmatrix} 0 & -9 \\ 4 & 0 \end{pmatrix}$$

$\mathbf{J}((0,0))$ has $\lambda_1 = 6$ and $\lambda_2 = -6$, giving a saddle point, with $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\mathbf{J}((3,2))$ has $\lambda_\pm = \pm 6i$, giving a center in the linearization.

The saddle structure persists in the nonlinear system, but this is not necessarily so for the center.

We can argue there is indeed a center at $(3,2)$ by considering that phase plane trajectories satisfy

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(2x-6)y}{(6-3y)x},$$

which is a separable equation for $y(x)$ yielding

$$\begin{aligned}\int \frac{6-3y}{y}dy &= \int \frac{2x-6}{x}dx \\ 6\ln(y) - 3y &= 2x - 6\ln(x) - C \\ \implies \qquad C &= 2x - 6\ln(x) + 3y - 6\ln(y) \\ C &= p(x) + q(y),\end{aligned}$$

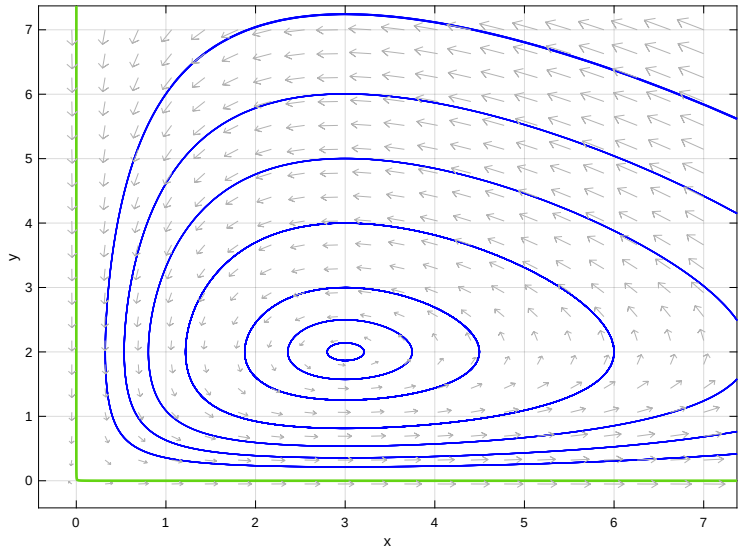
where $p(x) := 2x - 6\ln(x)$ and $q(y) := 3y - 6\ln(y)$. Hence, trajectories are level sets of the function

$$G(x, y) = p(x) + q(y)$$

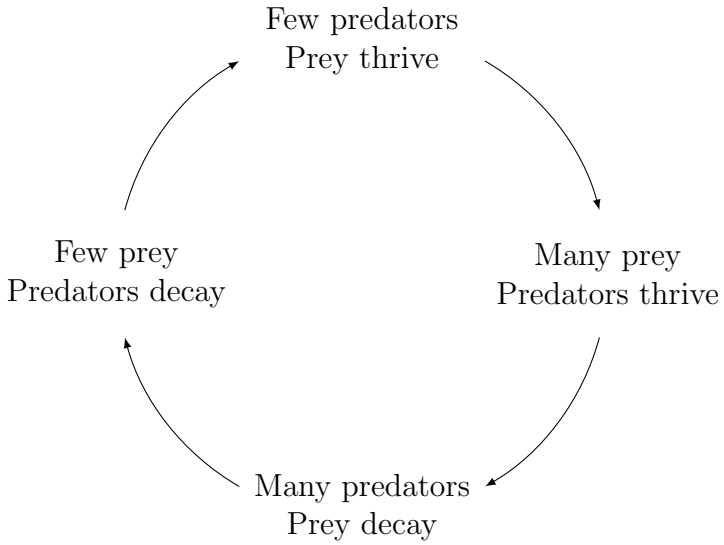
Noting that $p(x)$ is minimized at $x = 3$, and $q(y)$ at $y = 2$, we see that the minimum of G is (3,2).

As in the pendulum example, this makes trajectories closed loops: see this Geogabra applet.

This information can now be used to sketch the system's phase portrait.



All trajectories are closed loops, corresponding to (phase-shifted) oscillations in both predator and prey populations. These trajectories quantify the following population cycle:



□

These diagrams and conclusions are entirely generic of the predator-prey system (32).

Example: A damped pendulum with damping coefficient γ is described by

$$\theta'' + \gamma\theta' + \frac{g}{L}\sin(\theta) = 0.$$

Setting $\omega^2 := \frac{g}{L}$ and taking $\vec{\mathbf{x}} := \begin{pmatrix} \theta \\ \theta' \end{pmatrix}$ like before, this gives

$$\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) = \begin{pmatrix} y \\ -\omega^2 \sin(x) - \gamma y \end{pmatrix}.$$

The critical points $\vec{\mathbf{x}}_* = (n\pi, 0)$ are unchanged, but the linearizations are now given by

$$\vec{\mathbf{u}}\,' = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1}\omega^2 & -\gamma \end{pmatrix} \vec{\mathbf{u}}.$$

For even n, the eigenvalues are given by

$$\lambda_{\text{even}} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}.$$

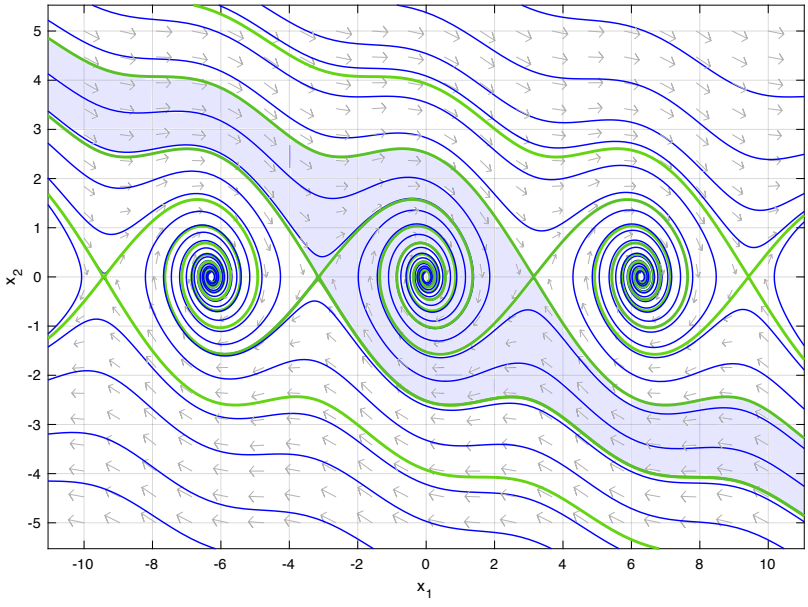
This is identical to a linear damped oscillator: it can be over-, under-, or critically damped based upon the sign of $\gamma^2 - 4\omega^2$, but the critical point is always asymptotically stable.

For odd n, the eigenvalues are

$$\lambda_{\text{odd}} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2},$$

which are always real with opposite sign, giving a saddle point.

Plotted below is a phase portrait representative of the underdamped case.



Green trajectories emanate from or terminate at saddle points (parallel to their eigenvectors).

Each stable spiral point has a basin of attraction (shaded for the origin): the set of points from which trajectories ultimately approach it. Basins are bounded by green trajectories, called separatrices.

We can no longer find an implicit formula for trajectories; $E(x, y) = \frac{1}{2}y^2 - \omega^2 \cos(x)$ now satisfies:

$$\begin{aligned}\frac{d}{dt}\left[E(x,y)\right] = \frac{d}{dt}\left[\frac{1}{2}y^2 - \omega^2\cos(x)\right] &= yy' + \omega^2\sin(x)x' \\ &= y(-\omega^2\sin(x) - \gamma y) + \omega^2\sin(x)y = -\gamma y^2,\end{aligned}$$

so that the energy function decreases along trajectories, as expected in the presence of damping. □

Day 38: Long-term behavior: Poincaré-Bendixson (5.4)

Where do system trajectories end up?

We have seen that phase plane trajectories can have a variety of behaviors. Let us see one more:

Example: Consider the system

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2), \\ y' &= x + y(1 - x^2 - y^2). \end{aligned}$$

Observe that the polar coordinates (r, θ) defined by $x = r\cos(\theta)$ and $y = r\sin(\theta)$ satisfy

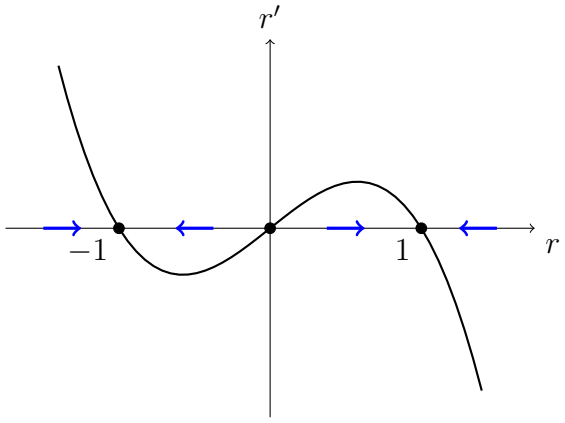
$$\begin{aligned} 2rr' = \frac{d}{dt}\left[r^2\right] &= \frac{d}{dt}\left[x^2 + y^2\right] \\ &= 2xx' + 2yy' \\ &= 2x\left[-y + x(1 - r^2)\right] + 2y\left[x + y(1 - r^2)\right] \\ &= 2r^2(1 - r^2) \\ \implies \qquad r' &= r(1 - r^2) \end{aligned}$$

and

$$\begin{aligned} \sec^2(\theta)\theta' = \frac{d}{dt}\left[\tan(\theta)\right] = \frac{d}{dt}\left[\frac{y}{x}\right] &= \frac{y'x - x'y}{x^2} \\ &= \frac{x\left[x + y(1 - r^2)\right] - y\left[-y + x(1 - r^2)\right]}{x^2} \\ &= \frac{r^2}{x^2} = \sec^2(\theta) \\ \implies \qquad \theta' &= 1. \end{aligned}$$

Hence $\theta(t)$ increases at a constant rate: phase plane trajectories orbit the origin indefinitely.

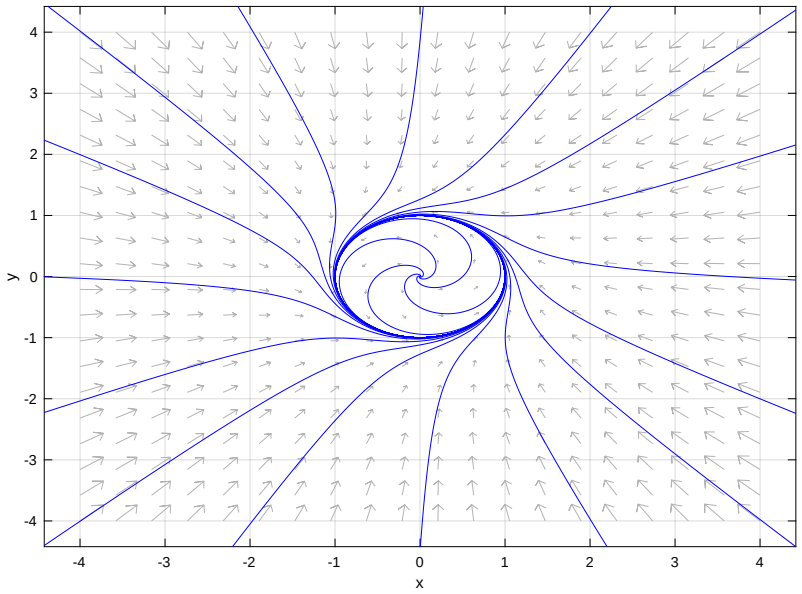
Meanwhile, $r(t)$ progresses according to $r' = r(1 - r^2)$, for which we may construct a phase line:



Only $r \geq 0$ is meaningful for the original problem: apparently, all nontrivial solutions have $r(t) \to 1$.

Trajectories in the phase plane, then, orbit the origin while approaching the circle $x^2 + y^2 = 1$.

The solution $\vec{\mathbf{x}}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$, which virtually all others approach, is called a limit cycle.



□

The Poincaré-Bendixson theorem classifies all the possible behaviors of trajectories as $t \to \infty$:

Theorem (Poincaré-Bendixson): The phase plane trajectory of a solution $\vec{\mathbf{x}}(t)$ to a 2D (differentiable) autonomous system $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ with isolated critical points must do one of a following:

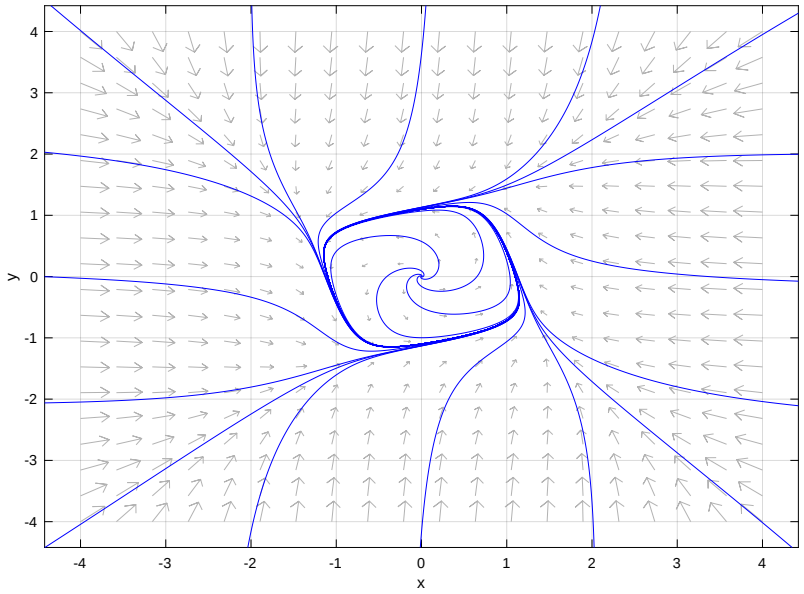
- be a critical point.
- approach a critical point as $t \to \infty$.
- form a closed loop.
- approach a limit cycle as $t \to \infty$.
- blow up to $\|\vec{\mathbf{x}}\| \to \infty$ (perhaps within finite time).

□

Example Consider the phase portrait shown below, for the system

$$x' = x - y - x^3,$$
$$y' = x + y - y^3.$$

Though we cannot write down a formula for it, the Poincaré-Bendixson theorem assures us that there is a limit cycle that the shown solutions approach as $t \to \infty$.



□

The Poincaré-Bendixson theorem tightly and neatly constrains behavior of 2D autonomous systems.

In $n \geq 3$ dimensions, one can have much richer structure, e.g. strange attractors and chaos.

Example: The Lorenz system, developed as a simplified model for atmospheric convection, is

$$\begin{aligned} x' &= \sigma(y - x), \\ y' &= x(\rho - z) - y, \\ z' &= xy - \beta z. \end{aligned}$$

It is a prototypical example of a system exhibiting chaotic behavior and admitting strange attractors (for certain parameter combinations σ, ρ, β).

A variety of interesting visuals can be found on this system's Wikipedia page.

The "strange attractor" to which almost all trajectories limit has fractal structure.

Nearby initial conditions may eventually diverge starkly– this is the phenomenon of chaos.

This system is often credited as the origin of the popular term "butterfly effect", not least because the shape of the strange attractor roughly resembles a pair of butterfly wings.

□