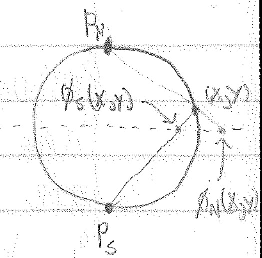
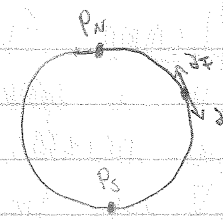
Day 2: Examples metric compatibility, torsion   09/04/19



Smooth Manifold example: $S^1 = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$.
Choose $P_N=(0,1)$, $P_S=(0,-1)$. We construct an atlas $U_N = S^1\setminus\{P_N\}$, $U_S = S^1\setminus\{P_S\}$ with $V_N = V_S = \mathbb{R}$ and $\phi_N : U_N \to V_N$ mapping $(x,y)\in S^1$ to the unique point of intersection of the line connecting (x,y) to P_N with the x-axis. The line in question has parameterization $(0,1)+(x,y-1)s = (sx,\ 1+(y-1)s)$, so it intersects the x-axis at $s=\frac{1}{1-y}$, i.e. the point $\left(\frac{x}{1-y},0\right)$, so $\phi_N(x,y)=\frac{x}{1-y}$. Similarly, $\phi_S(x,y)=\frac{x}{1+y}$. These are homeomorphisms, and since $\phi_N(x,y)\cdot\phi_S(x,y) = \frac{x^2}{1-y^2} = 1$, we have $\phi_N(x,y) = \frac{1}{\phi_S(x,y)}$, i.e. for $t\in V_S\setminus\{0\}$ $\phi_N\circ\phi_S^{-1}(t) = \frac{1}{t}$. Since this is smooth on its domain, the covering $\{U_N, U_S\}$ of S^1 by coordinate charts makes S^1 a smooth manifold. If we call the V_N coordinate $\bar t$, we have $\partial_t = \frac{\partial \bar t}{\partial t}\cdot\partial_{\bar t} = \frac{-1}{t^2}\cdot\partial_{\bar t}$.



Consider the curve $\gamma : (0,\infty)\to S^1$ given by $\gamma(s) = \left(\frac{2s}{1+s^2}, \frac{s^2-1}{1+s^2}\right)$, so
$t(s) = \phi_S(s) = \frac{1}{s}$; $\bar t(s) = \phi_N(s) = s$, and
γ naively appears nonaccelerating in V_N, but is certainly accelerating in V_S! $\dot\gamma(s) = -\frac{1}{s^2}\partial_t = \partial_{\bar t}$.
However, (this embedding of) S^1 has a natural connection induced by the ambient $\mathbb{R}^2$ in which the acceleration $\nabla_{\dot\gamma}\dot\gamma$ is given by the projection of $\gamma''(s)$ (as a vector in $\mathbb{R}^2$) onto the tangent space $T_{\gamma(s)}S^1$. For a general γ, then,

$$\nabla_{\dot\gamma}\dot\gamma \to \frac{\gamma''(s)\cdot(\gamma_2(s),-\gamma_1(s))}{\gamma_1(s)^2+\gamma_2(s)^2}(\gamma_2(s),-\gamma_1(s)) = (\gamma_1''\gamma_2-\gamma_2''\gamma_1)(\gamma_2(s),-\gamma_1(s))$$

The geodesic equation then reads
$0 = \gamma_1''(s)\gamma_2(s) - \gamma_2''(s)\gamma_1(s)$, together with $\gamma_1(s)^2+\gamma_2(s)^2 = 1$, which implies
$0 = \gamma_1(s)\gamma_1'(s) + \gamma_2(s)\gamma_2'(s)$
Solutions are $\gamma(s) = (\cos(s+\theta_0), \sin(s+\theta_0))$

What more can be said about metric compatibility and torsion? Given a reasonable physical restriction, they are quite related. Recall that we constructed the metric g by considering the local approximation of special Relativity — around any $p \in M$ there is a physical family of coordinate systems related via Lorentz transformations in which the preferred curves of gravity are straight lines. We required this approximation only truly hold "at p" to get g by requiring $g_p(\partial_i, \partial_j) = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$. We <u>might</u> ask that this approximation is a bit stronger "around p" by further requiring that $\partial_k(g_{ij})|_p = 0$. Under this constraint, we have the following:

<u>Lemma</u> Suppose that the preferred curves of gravity are geodesics of a connection ∇ on M and that around every $p \in M$ there exists an inertial coordinate system as above. Then the geodesics of ∇ are the geodesics of $\bar{\nabla}$, and ∇ is metric compatible iff its torsion tensor T is completely antisymmetric.

<u>Pf</u> Fix $p \in M$ and consider an inertial coordinate system at p. We interpret (rather minimalistically) the constraint that "the preferred curves of gravity are straight lines" in this coordinate system as $\Gamma_{ij}{}^k(p) = -\Gamma_{ji}{}^k(p)$ (so the geodesic equation reads $\ddot{\gamma}^k = 0$ at p). Then the metric compatibility tensor has components

$$M_{ijk}(p) = M_p(\partial_i, \partial_j, \partial_k) = [\partial_k(g_{ij}) - \Gamma_{ki}{}^l g_{lj} - \Gamma_{kj}{}^l g_{il}]|_p$$
$$= -(\Gamma_{ki}{}^l g_{lj} + \Gamma_{kj}{}^l g_{il})|_p$$

Since $T_{ijk}(p) = (\Gamma_{ij}{}^l - \Gamma_{ji}{}^l)\, g_{lk}|_p = 2\Gamma_{ij}{}^l g_{lk}|_p$, then we have

$$M_{ijk}(p) = -\tfrac{1}{2}(T_{kij}(p) + T_{kji}(p)).$$

Thus $M_p = 0$ iff T_p is antisymmetric in its second and third indices and hence iff T_p is completely antisymmetric.

Further, plugging $M_{ijk} = -\frac{1}{2}[T_{kij} + T_{kji}]$ into our formula for the difference tensor yields $D = \frac{1}{2}T$, so D is antisymmetric in its first two indices.