

Math 454 Lecture Notes

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Day 1: Introduction and terminology (Text: Lessons 1, 26)

Intro/Logistics (see Canvas syllabus):

- About me
- Course structure
- Grading policy
- Homework
- Prerequisites
- Office Hours: Tuesdays 1-2PM (EH 1832) & Thursdays 4-6PM (EH B737)

Recall that a differential equation is an equation containing derivatives of one or more unknown functions (dependent variables) with respect to one or more independent variables.

You have studied the case of one independent variable: ordinary differential equations (ODE's).

Example: A simple ODE for an unknown function $y(x)$ is $y'' + \lambda^2 y = 0$, with general solution

$$y(x) = A \cos(\lambda x) + B \sin(\lambda x).$$

Moving forward, one should be able to recognize such solutions reflexively.

□

We study the alternative: a partial differential equation (PDE) is one in which the unknown(s) depend on more than one variable, thereby involving partial derivatives.

Differential equations are essential to the modeling of physical phenomena.

Some famous PDE's fundamental to physics:

- (i) Schrodinger's equation: single-particle quantum mechanics.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V(\vec{x}, t) \Psi$$

Δ here is the Laplacian, $\Delta := \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

- (ii) Maxwell's equations: classical electricity and magnetism.

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho/\epsilon_0 & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{1}$$

(iii) Einstein's equation: general relativity.

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

It is helpful to introduce vocabulary classifying PDE's, characterizing:

- Order: the highest order derivative present in the equations.
- Dimension: number of independent variables (often, specifically spatial variables).
- Number of dependent variables (unknown functions).

Multiple unknowns $\implies$ multiple equations $\implies$ system of PDE's.

Systems of PDE's are generally beyond the scope of this course.

- Structure: a PDE is linear if the dependent variables and their derivatives only appear in a linear combination. E.g., a first-order linear PDE for the unknown $u(x, y)$ has the form

$$A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y). \quad (2)$$

Above,

- (i) is a second-order linear PDE, spatially 3D;
- (ii) is a first-order system of linear PDE's, spatially 3D;
- (iii) is a second-order system of nonlinear PDE's on 4D spacetime.

Let us see how some of the most important PDE's we'll study arise from the system (1).

Example: In *electrostatics*, all charges are at rest and fields have settled to a steady state.

This means $\vec{E}_t = \vec{B}_t = \vec{J} = \vec{0}$, so that the system (1) reduces to

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho/\epsilon_0 & \vec{\nabla} \times \vec{E} &= \vec{0} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{B} &= \vec{0} \end{aligned}$$

The last two of these imply that $\vec{B} = \vec{0}$ under physical boundary conditions, while $\vec{\nabla} \times \vec{E} = \vec{0}$ is equivalent¹ to $\vec{E} = \vec{\nabla}\phi$ for some function ϕ (electric potential). The remaining equation now reads

$$\Delta\phi = \rho/\epsilon_0.$$

This ubiquitous PDE is the Poisson equation. Away from charges, it reduces to Laplace's equation,

$$\Delta\phi = 0.$$

We will study both of these PDE's in detail.

□

¹When operating on a *simply connected* domain. Even when not, however, the implication still goes one way (the important way for physicists): solving $\Delta\phi = \rho/\epsilon_0$ for ϕ and taking $\vec{E} = \vec{\nabla}\phi$ will give a solution for $\vec{E}$.

Example: In *vacuum*, where there are no charges present, one has $\rho = 0$ and $\vec{\mathbf{J}} = \vec{\mathbf{0}}$, so

$$\begin{aligned}\vec{\nabla} \cdot \vec{\mathbf{E}} &= 0 & \vec{\nabla} \times \vec{\mathbf{E}} &= -\frac{\partial \vec{\mathbf{B}}}{\partial t} \\ \vec{\nabla} \cdot \vec{\mathbf{B}} &= 0 & \vec{\nabla} \times \vec{\mathbf{B}} &= \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}\end{aligned}$$

Taking the curl of the second equation and using the double-curl identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{E}}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{\mathbf{E}}) - \Delta \vec{\mathbf{E}},$$

one finds

$$\mu_0 \epsilon_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = \Delta \vec{\mathbf{E}}.$$

This is the wave equation (with speed $c := (\mu_0 \epsilon_0)^{-1/2}$), which we will also study in detail.

Similar manipulations yield the same equation for $\vec{\mathbf{B}}$.

□

As nearly all foundational techniques hinge upon linearity, we almost exclusively treat linear PDE's.

It is worthwhile, then, to characterize linearity more precisely.

First, a (partial) differential operator is a mapping of an input function to an output function, $u \mapsto O[u]$, involving (partial) derivatives of the input u .

Example: The mapping on functions $u(x, y)$ defined by

$$u \mapsto O[u] := u \cdot u_x \tag{3}$$

is a partial differential operator.

□

Definition: A (partial) differential operator L is called linear provided that

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2]$$

for any input functions u_1, u_2 and constants c_1, c_2 .

□

Note that the operator defined by (3) is nonlinear.

Definition: A DE is now called linear if it can be expressed in the form

$$L[u] = g$$

for some differential operator L and function g , often called the source.

A linear DE is further called homogeneous if $g = 0$, i.e. if it reads

$$L[u] = 0.$$

□

Example: The PDE (2) is linear with $g = D(x, y)$ and

$$L = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y} + C(x, y)$$

Indeed, this L is the most general first-order linear partial differential operator in two variables.

□

The key property making linear equations so immensely convenient is the principle of superposition:

Theorem: Let L be a linear differential operator. If u_1 and u_2 are, respectively, solutions to the linear DE's $L[u] = g_1$ and $L[u] = g_2$, then $\tilde{u} := c_1 u_1 + c_2 u_2$ satisfies the DE

$$L[\tilde{u}] = c_1 g_1 + c_2 g_2.$$

Proof: By linearity of the operator L ,

$$L[\tilde{u}] = L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2] = c_1 g_1 + c_2 g_2.$$

□

In particular, if u_1 and u_2 both satisfy $L[u] = 0$, then so does *every* linear combination $\tilde{u}$.

The most general second-order linear PDE in two variables (x, y) reads

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where each of A, B, C, D, E, F , and G may be functions of (x, y) .

Such equations will be the primary focus of this course. There are three basic categories:

- (a) Parabolic ($B^2 - 4AC = 0$). Describe heat flow and diffusion.
Prototype: $u_t = u_{xx}$ (heat equation).
- (b) Hyperbolic ($B^2 - 4AC > 0$). Describe vibrations and wave motion.
Prototype: $u_{tt} = u_{xx}$ (wave equation).
- (c) Elliptic ($B^2 - 4AC < 0$). Describe steady-state phenomena.
Prototype: $u_{xx} + u_{yy} = 0$ (Laplace's equation).

Day 2: Method of Characteristics (Lesson 27)

How can we construct solutions to first-order linear PDE's?

Recall: a first-order linear PDE in two variables has the form (2). With a slight relabelling, this is

$$A(x, t)u_x + B(x, t)u_t + C(x, t)u = D(x, t). \quad (4)$$

To “solve” such problems, we must establish what this means. Recall that solving a first-order ODE

$$y' = f(t, y)$$

requires an initial condition $y(t_0) = y_0$ —solutions have an arbitrary constant’s worth of freedom.

What information must we provide to solve (4)? Let us consider a very simple example.

Example:

$$u_x + 2u_t = 0$$

What does this equation say? Setting $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, note that this equation is equivalent to

$$\vec{v} \cdot \vec{\nabla} u = 0.$$

That is, this equation states that the *directional derivative* of u along $\vec{v}$ is 0: there is a particular direction through the xt -plane along which solutions u do not change.

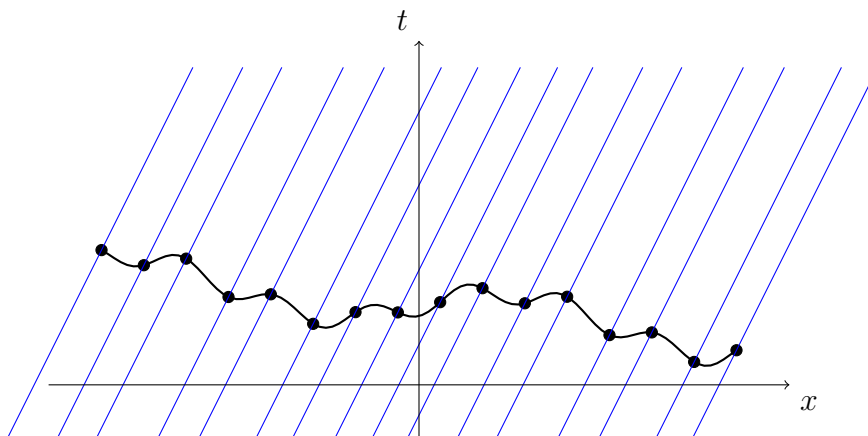
The PDE effectively propagates information along this direction. In particular, we have

$$u(x_0, t_0) = u_0 \quad \implies \quad u(x_0 + s, t_0 + 2s) = u_0 \quad \forall s \in \mathbb{R}.$$

If we foliate the xt -plane by lines with direction $\vec{v}$, called characteristics of the PDE, say

$$\vec{\gamma}_{x_0}(s) = (x_0 + s, 2s),$$

then specifying u at a single point along each characteristic $\vec{\gamma}_{x_0}$ will determine u everywhere.



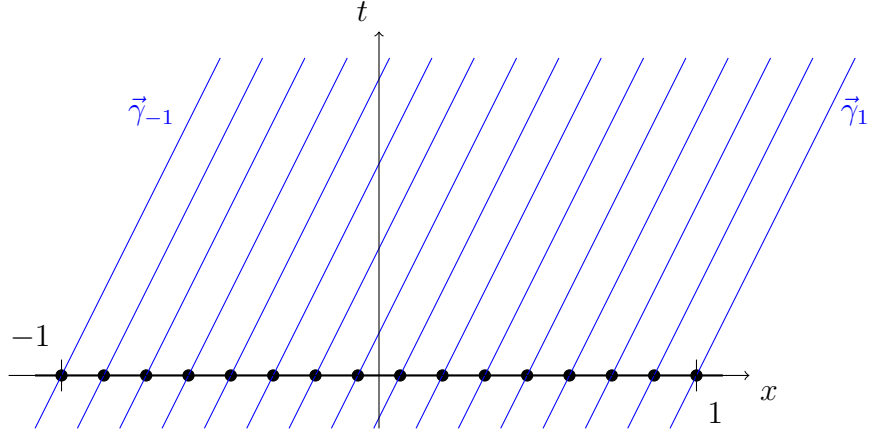
Information on the black curve propagates along the blue characteristics, determining u everywhere.

Apparently, then, we may naturally and meaningfully pose the initial value problem

$$u_x + 2u_t = 0, \quad u(x, 0) = \phi(x).$$

Indeed, the initial condition specifies u on each characteristic $\vec{\gamma}_{x_0}(s)$ at $s = 0$:

$$u(\vec{\gamma}_{x_0}(0)) = u(x_0, 0) = \phi(x_0).$$

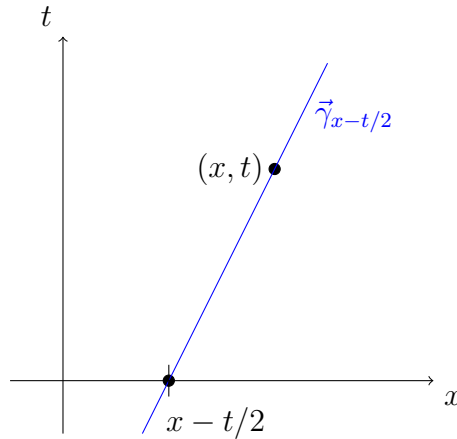


The schematic for propagating the information in a standard IVP for this PDE.

Let us solve for $u(x, t)$. We fix the point (x, t) and find the $\vec{\gamma}_{x_0}$ which passes through it:

$$(x, t) = \vec{\gamma}_{x_0}(s) = (x_0 + s, 2s) \implies s = t/2, \quad x_0 = x - t/2,$$

so the characteristic $\vec{\gamma}_{x-t/2}$ passes through (x, t) at $s = t/2$.



Hence we have

$$u(x, t) = u(\vec{\gamma}_{x-t/2}(t/2)) = u(\vec{\gamma}_{x-t/2}(0)) = \boxed{\phi(x - t/2)}$$

Observing that ϕ is arbitrary, we call $u(x, t) = \phi(x - t/2)$ the general solution to this PDE.

□

While ODE solutions allow arbitrary constants, PDE solutions allow arbitrary functions.

This example was simple. Mild complications in this same scheme arise for the general problem (4).

Let us add one more layer of complexity:

Example: Solve the IVP

$$u_x + 2u_t + u = x, \quad u(x, 0) = \phi(x).$$

Like before, and again setting $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we can recast in terms of a directional derivative:

$$\vec{v} \cdot \vec{\nabla} u + u = x.$$

The problem apparently only involves a single derivative: morally, this is an ODE in the $\vec{v}$ direction.

To make this more than moral, fix x_0 and consider the function $f(s) := (u \circ \vec{\gamma}_{x_0})(s)$. Then

$$\begin{aligned} f'(s) &= \vec{\gamma}'_{x_0}(s) \cdot (\vec{\nabla} u \circ \vec{\gamma}_{x_0})(s) \\ &= (\vec{v} \cdot \vec{\nabla} u) \circ \vec{\gamma}_{x_0}(s) \\ &= (x - u) \circ \vec{\gamma}_{x_0}(s) \\ &= x_0 + s - f(s) \end{aligned}$$

That is, $f(s)$ satisfies the simple ODE

$$f'(s) + f(s) = x_0 + s,$$

to which the general solution (using, say, the integrating factor e^s) is

$$f(s) = x_0 + s - 1 + Ce^{-s}.$$

To determine C , note that

$$x_0 - 1 + C = f(0) = u(\vec{\gamma}_{x_0}(0)) = u(x_0, 0) = \phi(x_0),$$

requiring $C = \phi(x_0) + 1 - x_0$, and hence

$$f(s) = x_0 + s - 1 + (\phi(x_0) + 1 - x_0)e^{-s}.$$

Information propagates along characteristics as before; this computation tells us *how* it propagates.

In the previous example, the propagation was trivial: $(u \circ \vec{\gamma}_{x_0})(s)$ was simply constant. More generally, it is a function of s determined by a first-order linear ODE and the initial conditions.

We may now determine $u(x, t)$ in much the same way as before, using $(x, t) = \gamma_{x-t/2}(t/2)$:

$$\begin{aligned} u(x, t) &= u(\vec{\gamma}_{x-t/2}(t/2)) = (u \circ \vec{\gamma}_{x-t/2})(t/2) \\ &= (x - t/2) + t/2 - 1 + [\phi(x - t/2) + 1 - (x - t/2)]e^{-t/2} \\ &= \boxed{x - 1 + [\phi(x - t/2) + 1 - x + t/2]e^{-t/2}} \end{aligned}$$

Note that we've simply substituted $x_0 = x - t/2$ and $s = t/2$ into our solution for $f(s)$.

□

The final step is to allow variable coefficients to u_x and u_t , adding still more complexity:

Example: Solve the IVP

$$tu_x + xu_t + u = x, \quad u(x, 0) = \phi(x) \text{ for } x > 0.$$

We proceed in much the same way, recasting in terms of a directional derivative,

$$\vec{\mathbf{v}} \cdot \vec{\nabla} u + u = x,$$

except that the direction $\vec{\mathbf{v}}(x, t) = \begin{pmatrix} t \\ x \end{pmatrix}$ is no longer constant. Information still propagates in the direction of $\vec{\mathbf{v}}$, i.e. along the characteristic curves $\vec{\gamma}_{x_0}(s)$ everywhere tangent to $\vec{\mathbf{v}}$, defined by:

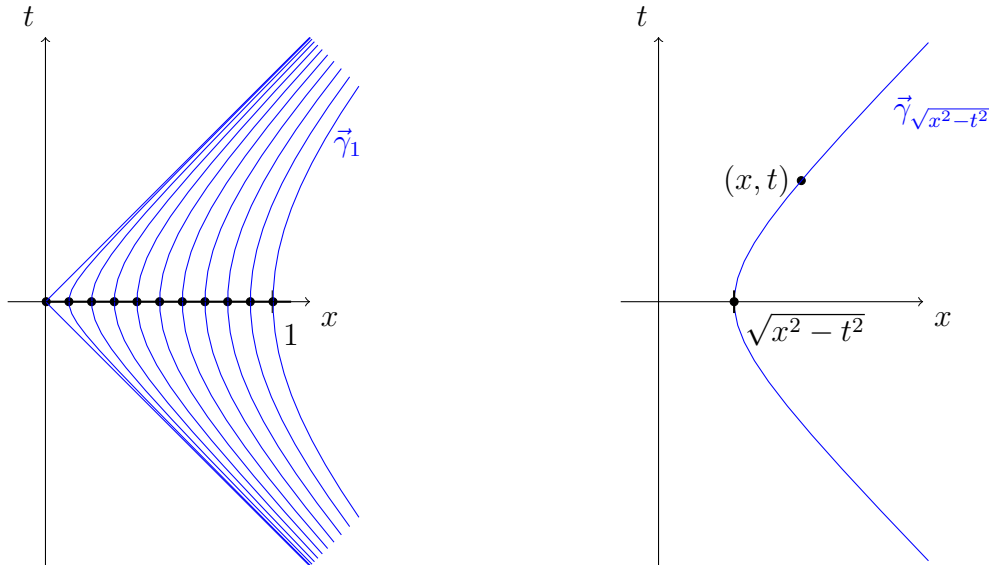
$$\vec{\gamma}'_{x_0}(s) = \vec{\mathbf{v}} \circ \vec{\gamma}_{x_0}(s), \quad \vec{\gamma}_{x_0}(0) = (x_0, 0).$$

More familiarly: the coordinates of $\vec{\gamma}_{x_0}(s) = (x(s), t(s))$ are determined by the ODE system IVP²,

$$\begin{aligned} x'(s) &= v_x = t, & x(0) &= x_0, \\ t'(s) &= v_t = x, & t(0) &= 0. \end{aligned}$$

The solution to this system is $x(s) = x_0 \cosh(s)$, $t(s) = x_0 \sinh(s)$, so that

$$\vec{\gamma}_{x_0}(s) = (x_0 \cosh(s), x_0 \sinh(s)).$$



The schematic propagation of initial data in this IVP.

According to the characteristics, our initial data can only propagate to those (x, t) with $|t| < x$.

²Note that v_x and v_t here denote the x and t components of the vector $\vec{\mathbf{v}}$, not partial derivatives. Subscripts on the un-bolded form of a vector variable will always refer to components in this way.

We now proceed as in the previous example, but using this $\vec{\gamma}_{x_0}$. Now $f(s) := (u \circ \vec{\gamma}_{x_0})(s)$ satisfies

$$\begin{aligned} f'(s) &= \vec{\gamma}'_{x_0}(s) \cdot (\vec{\nabla} u \circ \vec{\gamma}_{x_0})(s) \\ &= (\vec{v} \cdot \vec{\nabla} u) \circ \vec{\gamma}_{x_0}(s) \\ &= (x - u) \circ \vec{\gamma}_{x_0}(s) \\ &= x_0 \cosh(s) - f(s), \end{aligned}$$

so that

$$f'(s) + f(s) = x_0 \cosh(s),$$

Again using the integrating factor e^s , one has

$$f(s) = x_0 e^{-s} \cdot \int e^s \cosh(s) ds = x_0 e^{-s} \cdot \int \frac{e^{2s} + 1}{2} ds = \frac{x_0}{4} (e^s + 2s e^{-s}) + C e^{-s},$$

and setting

$$\frac{x_0}{4} + C = f(0) = u(\vec{\gamma}_{x_0}(0)) = u(x_0, 0) = \phi(x_0)$$

yields $C = \phi(x_0) - \frac{x_0}{4}$, and hence

$$f(s) = \frac{x_0}{4} [e^s + (2s - 1)e^{-s}] + \phi(x_0)e^{-s}.$$

To solve for $u(x, t)$, we must again find the characteristic which passes through it:

$$(x, t) = \vec{\gamma}_{x_0}(s) = (x_0 \cosh(s), x_0 \sinh(s)) \implies s = \tanh^{-1}(t/x), \quad x_0 = \sqrt{x^2 - t^2}.$$

We now simply plug these expressions³ for s and x_0 into our solution for $f(s)$:

$$\begin{aligned} u(x, t) &= \frac{\sqrt{x^2 - t^2}}{4} \left[\sqrt{\frac{x+t}{x-t}} + (2 \tanh^{-1}(t/x) - 1) \sqrt{\frac{x-t}{x+t}} \right] + \sqrt{\frac{x-t}{x+t}} \phi(\sqrt{x^2 - t^2}) \\ &= \frac{1}{4} \left[|x+t| + \left(\ln \left(\frac{x+t}{x-t} \right) - 1 \right) |x-t| \right] + \sqrt{\frac{x-t}{x+t}} \phi(\sqrt{x^2 - t^2}) \\ &= \boxed{\frac{1}{4} \left[2t + (x-t) \ln \left(\frac{x+t}{x-t} \right) \right] + \sqrt{\frac{x-t}{x+t}} \phi(\sqrt{x^2 - t^2})} \end{aligned}$$

on the region $\{(x, t) : 0 \leq |t| < x\}$.

□

One can see these examples' evolutions of initial data by playing the t slider in [this](#) Desmos demo.

In these examples, we see the general procedure emerge (for an IVP specifying $u(x, 0)$):

1. Find a formula for $\vec{\gamma}_{x_0}(s) = (x(s), t(s))$ by solving the ODE system

$$\begin{aligned} x'(s) &= v_x, & x(0) &= x_0, \\ t'(s) &= v_t, & t(0) &= 0. \end{aligned}$$

³Note that $\tanh^{-1}(t/x) = \ln \left(\sqrt{\frac{x+t}{x-t}} \right)$

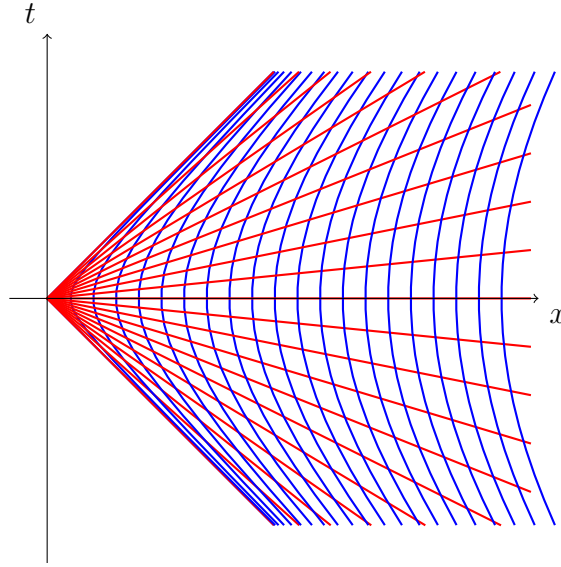
2. Find the ODE characterizing $f(s) := (u \circ \vec{\gamma}_{x_0})(s)$ and solve for $f(s)$ in terms of s and x_0 .

3. Find the combination of x_0 and s such that $\vec{\gamma}_{x_0}(s) = (x, t)$ and plug into $f(s)$ to find $u(x, t)$.

In general, step 1 may need to be minorly adapted to the initial data (e.g. if $u(x, 1) = \phi(x)$ is given).

Note: the text frames this method as a change of coordinates under which the PDE becomes an ODE; this is simply a matter of perspective. In our notation, the new coordinates would be (x_0, s) , and the transformation between them and the original coordinates (x, t) is simply

$$(x_0, s) \mapsto (x, t) = \vec{\gamma}_{x_0}(s).$$



The (x_0, s) coordinate grid from the final example. Blue curves have fixed x_0 , while red curves have fixed s .

Day 3: Heat Equation Intro (Lessons 2,4)

How can we model the dynamics of temperature?

In ODE's, we (likely) saw *Newton's law of cooling*: the rate of change of an object's temperature is proportional to its temperature difference with the environment. If the temperature is $y(t)$,

$$y'(t) = -k(y(t) - T). \quad (5)$$

About as well as one can do with a single independent variable (time), but quite crude.

If a region's temperature $u(\vec{x}, t)$ varies spatially *and* temporally, what governs its evolution?

Requires *Fourier's law*: vectorial heat flux $\vec{q}$ ($\frac{(\text{energy})}{(\text{time}) \cdot (\text{area})}$) is proportional to temperature gradient,

$$\vec{q} = -k \vec{\nabla} u.$$

Physically, the rate that heat energy crosses an infinitesimal surface area element $d\vec{A}$ is

$$\vec{q} \cdot d\vec{A} = -k \vec{\nabla} u \cdot d\vec{A}.$$

The quantity k ($\frac{(\text{energy})}{(\text{temperature}) \cdot (\text{time}) \cdot (\text{length})}$) is the medium's thermal conductivity. Equivalently,

$$(\text{heat flux in direction } \hat{n}) = \hat{n} \cdot \vec{q} = -k \hat{n} \cdot \vec{\nabla} u,$$

proportional to the temperature's rate of change in the $\hat{n}$ direction. Since this rate of change is

$$\hat{n} \cdot \vec{\nabla} u \Big|_{\vec{x}_0} = \frac{d}{ds} [u(\vec{x}_0 + s\hat{n}, t)] \Big|_{s=0} = \lim_{h \rightarrow 0} \frac{u(\vec{x}_0 + h\hat{n}, t) - u(\vec{x}_0, t)}{h},$$

this intuitively jives with Newton's law: heat flow along $\hat{n}$ arises due to a temperature difference.

What does Fourier's Law imply about temperature dynamics?

Consider the total heat energy in a bounded 3D region $U \subset \mathbb{R}^3$ (with smooth boundary ∂U):

$$H(t) = \int_U c \rho u(\vec{x}, t) dV,$$

with c the medium's specific heat ($\frac{(\text{energy})}{(\text{temperature}) \cdot (\text{mass})}$) and ρ its mass density. On one hand⁴,

$$\frac{dH}{dt} = \frac{d}{dt} \left[\int_U c \rho u(\vec{x}, t) dV \right] = \int_U c \rho u_t(\vec{x}, t) dV.$$

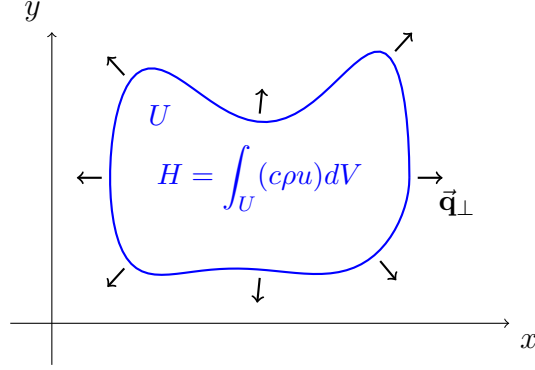
But since energy is conserved: if energy isn't generated in U , the change must flow across ∂U ⁵,

$$\frac{dH}{dt} = - \int_{\partial U} \vec{q} \cdot d\vec{A} = \int_{\partial U} k \vec{\nabla} u \cdot d\vec{A} = \int_U (k \Delta u) dV \quad (6)$$

⁴A mathematician worries whether interchanging the derivative and integral is permissible. This is generally a nontrivial analysis question, but it goes through here since U is bounded and we take u to be smooth.

⁵Recall the *divergence theorem*:

$$\int_{\partial U} \vec{F} \cdot d\vec{A} = \int_U (\vec{\nabla} \cdot \vec{F}) dV$$



Hence we've found that

$$\int_U [c\rho u_t - k\Delta u] dV = 0. \quad (7)$$

Applies to any bounded region U , no matter how large, small, or (smoothly) misshapen.

Since the integral (7) is 0 for *all* such U , the integrand ($c\rho u_t - k\Delta u$) must be 0⁶:

$$c\rho u_t - k\Delta u = 0.$$

Rearranging and setting $\alpha^2 := \frac{k}{c\rho}$ (thermal diffusivity), we've arrived at the heat equation:

$$\boxed{u_t = \alpha^2 \Delta u} \quad (8)$$

This PDE governs the dynamics of temperature in a uniform medium.

Note that this PDE is linear and homogeneous, $L[u] = 0$, with $L = (\partial_t - \alpha^2 \Delta)$. Some variations:

- Heat sources: if heat *is* generated in U at density rate $f(\vec{x}, t)$ ($\frac{\text{(energy)}}{\text{(volume)} \cdot \text{(time)}}$), then (6) becomes

$$\frac{dH}{dt} = \int_U (k\Delta u + f(\vec{x}, t)) dV,$$

and (8) is now

$$\boxed{u_t = \alpha^2 \Delta u + F(\vec{x}, t)} \quad (9)$$

with $F(\vec{x}, t) = \frac{1}{c\rho} f(\vec{x}, t)$. This reads $L[u] = F$: heat sources add an inhomogeneous term.

- *Fick's Law of diffusion*: the flux $\vec{J}$ of diffusing particles is proportional to the density gradient,

$$\vec{J} = -D \vec{\nabla} \rho,$$

with D the diffusion coefficient. A relabelling of variables in the above argument yields

$$\rho_t = D \Delta \rho$$

(using *conservation of particle number*), so the heat equation also describes diffusion.

⁶This follows so long as the integrand is (reasonably) assumed continuous.

- Convection: if the medium is flowing with constant velocity $\vec{v}$, heat energy (or particle number) in U can also change due to the medium's flow across the boundary,

$$(\text{rate energy is carried out of } U) = \int_{\partial U} c\rho u \vec{v} \cdot \mathbf{d}\vec{A} = \int_U c\rho \vec{\nabla} \cdot (u \vec{v}) dV = \int_U (c\rho \vec{v} \cdot \vec{\nabla} u) dV$$

Hence (6) gains this as an additional term,

$$\frac{dH}{dt} = \int_U (k\Delta u - c\rho \vec{v} \cdot \vec{\nabla} u) dV,$$

and ultimately (8) becomes the convection-diffusion equation

$$\boxed{u_t + \vec{v} \cdot \vec{\nabla} u = \alpha^2 \Delta u} \quad (10)$$

From Day 2: if $\alpha \rightarrow 0$, this simply translates initial data at velocity $\vec{v}$, as one would expect.

- In a *nonuniform* medium, k may vary spatially. In a *nonlinear* medium, k may depend on the temperature u . You will derive the appropriate modifications to (8) in HW1.
- 1D: In a thin rod, u is nearly constant on cross-sections, so $u(\vec{x}, t) \rightarrow u(x, t)$ and (8) becomes

$$u_t = \alpha^2 u_{xx}, \quad (11)$$

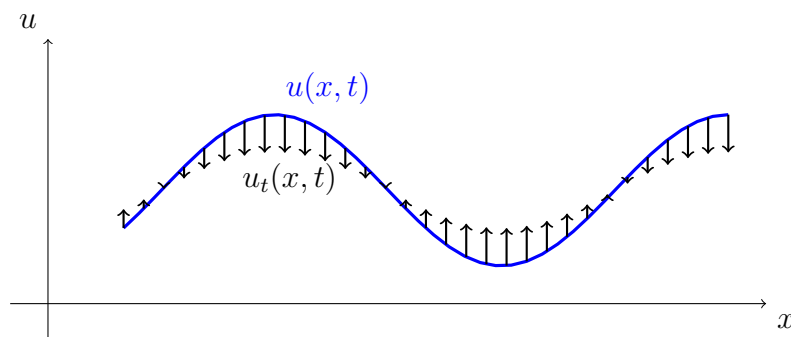
the 1D heat equation. Observe that this says

$$u_t \propto u_{xx} = \lim_{h \rightarrow 0} \frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2}$$

$$u_t \propto \lim_{h \rightarrow 0} -\frac{2}{h^2} \left[u(x, t) - \frac{u(x+h, t) + u(x-h, t)}{2} \right].$$

In brackets is the difference between $u(x, t)$ and its *mean* value at nearby points⁷. Compare (5).

Graphically, (11) says that temperature changes more quickly in response to strong *concavity*.



⁷In fact, denoting by $\bar{u}$ the average of u on a sphere of radius r centered at $\vec{x}$, one has in any dimension n that

$$\bar{u} = u(\vec{x}) + \frac{\Delta u(\vec{x})}{2n} r^2 + O(r^3).$$

In this sense, Δ measures a function's deviation from its mean, and the heat equation reflects Newton's law of cooling.

A new consideration arises in 1D: does the rod lose heat to its 3D environment? If not, we say it is laterally insulated, and (11) applies. If so, Newton's law suggests a simple correction,

$$u_t = \alpha^2 u_{xx} - \beta(u - u_0),$$

where u_0 is the environment temperature.

In practice, one may need several of the above variations' correction terms simultaneously.

Day 4: IBVP's and Separation of Variables (Lessons 2,3,5)

How do we pose temperature evolution problems?

Given our PDE, say (8), what more must we specify to uniquely determine a solution $u(\vec{x}, t)$?

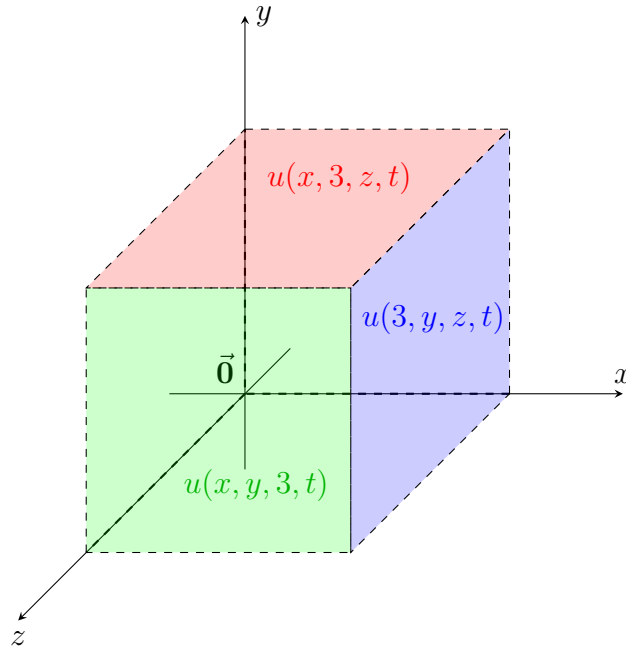
Intuitively: given a cubic room, say $0 < x, y, z < 3$, what must we know to predict its temperature?

Initial data $u(\vec{x}, 0)$ is necessary, but insufficient. Are there windows? Is it cold or warm outside?

Namely, we need conditions at the *boundary* of our room. We can't hope to predict $u(\vec{x}, t)$ otherwise.

In addition to the PDE (8) and initial condition $u(\vec{x}, 0)$, then, we must constrain the six functions

$$\begin{array}{lll} u(x, y, 0, t) & \text{for } 0 < x, y < 3, & u(x, 0, z, t) & \text{for } 0 < x, z < 3, & u(0, y, z, t) & \text{for } 0 < y, z < 3. \\ u(x, y, 3, t) & & u(x, 3, z, t) & & u(3, y, z, t) & \end{array}$$



One can imagine placing sensors on a boundary (say, the front surface) and applying heating elements as needed to enforce a desired boundary temperature. This amounts to directly specifying

$$u(x, y, 3, t) = g(x, y, t).$$

Another surface (say, the top) may be thickly covered in insulating material, so virtually no heat transfers across it. This requires $\vec{q} \cdot \hat{n} = 0$, with $\hat{n}$ the boundary's unit normal. By Fourier's law,

$$\frac{\partial u}{\partial n}(x, 3, z, t) = u_y(x, 3, z, t) = 0.$$

Alternatively, heating elements may be set up to transfer heat at a prescribed rate, specifying

$$\frac{\partial u}{\partial n}(x, 3, z, t) = u_y(x, 3, z, t) = g(x, 3, z, t).$$

Finally, a third surface (say, the rightmost) may be a glass window exposed to an environment temperature $u_0(t)$. Newton's law suggests that $\vec{q} \cdot \hat{n}$ should be proportional to the difference $u - u_0$:

$$k \frac{\partial u}{\partial n}(3, y, z, t) = -h [u(3, y, z, t) - u_0(t)],$$

with h the boundary's heat-exchange coefficient.

In summary, the most common and natural boundary conditions are:

- $u = g$ (specified boundary temperature).
- $\frac{\partial u}{\partial n} = g$ (heat flow across boundary specified).
- $\frac{\partial u}{\partial n} + \lambda u = g$ (surrounding environment temperature is specified).

Example: Consider a laterally insulated copper rod situated at $0 < x < 200$ (measured in cm) with an initial temperature of 0°C . Suppose the end at $x = 0$ is perfectly insulated, while the end at $x = 200$ is immersed in water at 20°C . The rod's temperature $u(x, t)$ satisfies

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \\ u_x(0, t) &= 0, \\ u_x(200, t) &= -\frac{h}{k} [u(200, t) - 20], \quad 0 \leq t < \infty \\ u(x, 0) &= 0, \quad 0 < x < 200 \end{aligned}$$

For copper, $\alpha^2 \approx 1.16 \frac{\text{cm}^2}{\text{s}}$ and $k \approx 0.93 \frac{\text{cal}}{\text{cm-s-C}}$. h ($\frac{\text{cal}}{\text{s-C}}$) generally must be found empirically. □

This is a typical initial boundary value problem (IBVP) containing all the information needed to obtain a unique solution in parabolic-type problems.

Similarly to DE's, we will call a boundary condition linear if it has the form

$$\alpha u + \beta \frac{\partial u}{\partial n} = g$$

(with α, β constants) and homogeneous if $g = 0$. Linear BC's also enjoy a superposition principle.

As seen in the above discussion, linear boundary conditions are apparently quite natural.

How can we construct solutions to IBVP's?

To solve these, we now develop one of the most powerful techniques for understanding linear PDE's.

Example: We model a laterally insulated rod of length 1 with temperature fixed to 0 at the ends.

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \\ u(0, t) &= 0, \\ u(1, t) &= 0, \quad 0 \leq t < \infty, \\ u(x, 0) &= \sin(\pi x), \quad 0 \leq x \leq 1. \end{aligned} \tag{12}$$

An interesting observation about this initial condition:

$$u_{xx}(x, 0) = -\pi^2 \sin(\pi x) = -\pi^2 u(x, 0),$$

so that, at least at $t = 0$, u seems to satisfy the simple ODE in t ,

$$u_t = -(\pi\alpha)^2 u. \quad (13)$$

Not too informative at a single t . If it (miraculously) holds at *all* t , one has the simple solution

$$u(x, t) = u(x, 0)e^{-(\pi\alpha)^2 t} = \sin(\pi x)e^{-(\pi\alpha)^2 t}.$$

Remarkably: this function $u(x, t)$ keeps $u_{xx} = -\pi^2 u$ consistent, so

$$\boxed{u(x, t) = \sin(\pi x)e^{-(\pi\alpha)^2 t}}$$

is a solution to (12) for which (13) *does* always hold.

The essential feature that preserved $u_{xx} = -\pi^2 u$ was that the solution $u(x, t) = \sin(\pi x)e^{-(\pi\alpha)^2 t}$ *separated* into a product of a function of x and a function of t . □

In general, the rod above may have an arbitrary initial temperature profile $u(x, 0) = \phi(x)$:

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \\ u(0, t) &= 0, \\ u(1, t) &= 0, \quad 0 \leq t < \infty, \\ u(x, 0) &= \phi(x), \quad 0 \leq x \leq 1. \end{aligned} \quad (14)$$

We eventually expect $u(x, t) \rightarrow 0$. How does the transition from $\phi(x)$ look? How long does it take?

This requires knowing $u(x, t)$. Generally, solutions may be quite complicated.

The example suggests we might reduce to ODE's by seeking solutions which separate,

$$u(x, t) = X(x)T(t).$$

This is the separation of variables ansatz. Plugging this ansatz into the PDE gives

$$X(x)T'(t) = u_t = \alpha^2 u_{xx} = \alpha^2 X''(x)T(t).$$

The crucial step to fully “separate” the variables in the PDE is dividing through by $X(x)T(t)$,

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$$

This equation states that the LHS and RHS are the same function of (x, t) . From the LHS, this function is independent of x . From the RHS, it is independent of t . Hence it must be *constant*! Say,

$$\frac{T'(t)}{\alpha^2 T(t)} = k = \frac{X''(x)}{X(x)}.$$

A priori, $k \in \mathbb{R}$ may be any number. This yields two ODE's for $X(x)$ and $T(t)$,

$$\begin{aligned} X'' - kX &= 0, \\ T' - k\alpha^2 T &= 0. \end{aligned}$$

Solving, these ODE's indicate that any function of the forms

$$\begin{aligned} u(x, t) &= \left[A \cosh(\sqrt{k}x) + B \sinh(\sqrt{k}x) \right] e^{k\alpha^2 t}, & k > 0 \\ u(x, t) &= \left[A \cos(\sqrt{-k}x) + B \sin(\sqrt{-k}x) \right] e^{k\alpha^2 t}, & k < 0 \end{aligned}$$

solves our PDE. How many also satisfy the boundary conditions $u(0, t) = u(1, t) = 0$?

These boundary conditions require $X(0) = X(1) = 0$. Recall from HW0 that the BVP

$$y'' - \lambda^2 y = 0, \quad y(0) = y(1) = 0$$

only admits $y(x) = 0$. To find a nontrivial $X(x)$, then, we must take $k < 0$ ⁸, say $k \rightarrow -\lambda^2$:

$$\begin{aligned} X'' + \lambda^2 X &= 0, & X(0) = X(1) &= 0, \\ T' + (\lambda\alpha)^2 T &= 0. \end{aligned} \tag{15}$$

For which λ does the BVP for $X(x)$ admit nontrivial solutions? As seen above, one must have

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x),$$

and imposing $0 = X(0) = A$ and $0 = X(1) = B \sin(\lambda)$ requires $\sin(\lambda) = 0$, or $\lambda = n\pi$ for $n \in \mathbb{Z}$.

Hence for each $\lambda = n\pi$, $n \in \mathbb{Z}_+$, we have $X_n(x) = B \sin(n\pi x)$. Correspondingly, $T_n(t) = e^{-(\pi n\alpha)^2 t}$.

For each n , then, we have a solution $u_n(x, t)$ to the PDE respecting the boundary conditions,

$$u_n(x, t) = B_n \sin(n\pi x) e^{-(\pi n\alpha)^2 t}.$$

These are *all* of the separable solutions to the PDE matching the boundary conditions.

⁸Alternatively, we might take $k < 0$ on physical grounds since we expect $u(x, t) \rightarrow 0$ at large t .

Day 5: Separation of Variables and Fourier Series (Lessons 5,11)

How can we leverage separable BVP solutions to solve any IBVP?

We saw above that all separable solutions to the BVP in (14) were

$$u_n(x, t) = B_n \sin(n\pi x) e^{-(n\pi\alpha)^2 t}.$$

This is just dandy if $u(x, 0) \propto \sin(n\pi x)$, but what does it have to do with an arbitrary $\phi(x)$?

Superposition now shines. Since both the PDE and BC's in (14) are homogeneous, if u_1 and u_2 each satisfy the BVP, so does any combination $c_1 u_1 + c_2 u_2$. More broadly, any combination

$$\sum_n B_n u_n(x, t) = \sum_n B_n \sin(n\pi x) e^{-(n\pi\alpha)^2 t} \quad (16)$$

will solve the BVP⁹. In the full IBVP (14), this allows us to match any initial condition of the form

$$\phi(x) = \sum_n B_n \sin(n\pi x) \quad (17)$$

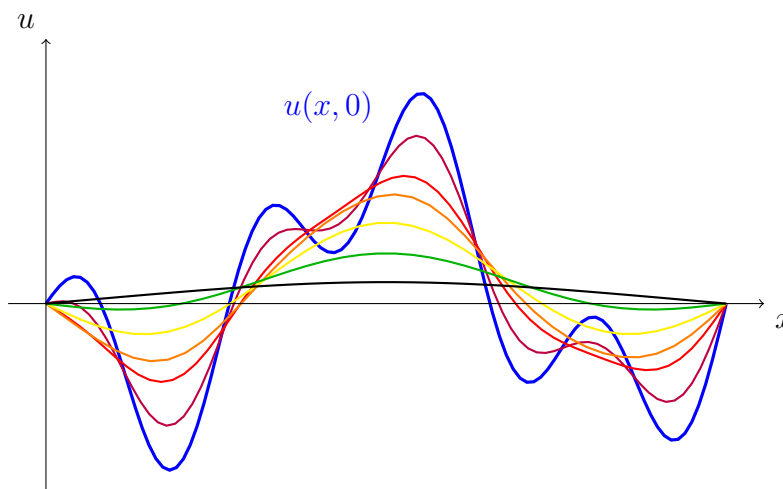
for any collection of constants B_n .

Example: The solution to the IBVP

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \\ u(0, t) &= 0, \\ u(1, t) &= 0, & 0 \leq t < \infty, \\ u(x, 0) &= \frac{1}{2} \sin(\pi x) - \frac{3}{2} \sin(3\pi x) + \sin(8\pi x), & 0 \leq x \leq 1. \end{aligned}$$

is simply

$$u(x, t) = \frac{1}{2} \sin(\pi x) e^{-(\pi\alpha)^2 t} - \frac{3}{2} \sin(3\pi x) e^{-(3\pi\alpha)^2 t} + \sin(8\pi x) e^{-(8\pi\alpha)^2 t}$$



⁹When verifying (16) satisfies the PDE $u_t = \alpha^2 u_{xx}$ as an *infinite* sum, a mathematician worries whether interchanging the derivatives and sum is permissible. This is generally a nontrivial analysis question which we won't pay much heed. In our setting, it will work due to the fast convergence of the sum at any $t > 0$, thanks to the $e^{-(n\pi\alpha)^2 t}$.

Some snapshots of this evolution are shown above. See also [this](#) Desmos demo.

□

It may seem that (17) is a rather restrictive assumption, but it was a remarkable insight of Fourier that there is a sense in which it covers *all* (well-behaved) functions $\phi(x)$ on $[0, 1]$.

To begin to see this, let us suppose that our function $\phi(x)$ could be expanded in this way,

$$\phi(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x). \quad (18)$$

How could we find the numbers B_n ?

This can be done by leveraging the orthogonality of the function family $\{\sin(n\pi x)\}$ on $(0, 1)$:

$$\int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \frac{1}{2}(1 - \delta_{n,0})\delta_{n,m} = \begin{cases} \frac{1}{2} & n = m > 0 \\ 0 & n = m = 0 \\ 0 & n \neq m \end{cases}$$

You will verify this in HW2. Multiplying (18) by $\sin(m\pi x)$ and integrating, we find¹⁰

$$\begin{aligned} \int_0^1 \phi(x) \sin(m\pi x) dx &= \int_0^1 \left[\sum_{n=1}^{\infty} B_n \sin(n\pi x) \sin(m\pi x) \right] dx \\ &= \sum_{n=1}^{\infty} B_n \left[\int_0^1 \sin(n\pi x) \sin(m\pi x) dx \right] = \sum_{n=1}^{\infty} \frac{1}{2} B_n \delta_{n,m} = \frac{1}{2} B_m \end{aligned}$$

That is, if the expansion (18) exists, the coefficients B_n *must* be given by

$$\boxed{B_n = 2 \int_0^1 \phi(x) \sin(n\pi x) dx} \quad (19)$$

The series

$$S(x) := \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

with coefficients (19) is the Fourier sine series for $\phi(x)$ on $0 \leq x \leq 1$.

Example: Consider the function $\phi(x) = x(1 - x)$. It is a straightforward procedure to compute

$$B_n = 2 \int_0^1 \phi(x) \sin(n\pi x) dx = 2 \int_0^1 (x - x^2) \sin(n\pi x) dx$$

¹⁰A mathematician worries whether interchanging the integral and the infinite sum is permissible. This is generally a nontrivial analysis question which we won't pay much heed. If this bothers you, it may bring you peace of mind to know that Lebesgue's dominated convergence theorem ensures that it works here so long as $\sum_{n=1}^{\infty} |B_n|$ converges.

$$\begin{aligned}
&= \frac{2}{n\pi} \left[-(x - x^2) \cos(n\pi x) \Big|_0^1 + \int_0^1 (1 - 2x) \cos(n\pi x) dx \right] \\
&= \frac{2}{(n\pi)^2} \left[(1 - 2x) \sin(n\pi x) \Big|_0^1 + 2 \int_0^1 \sin(n\pi x) dx \right] \\
&= \frac{4}{(n\pi)^3} [-\cos(n\pi x)] \Big|_0^1 = -\frac{4}{(n\pi)^3} [\cos(n\pi) - 1] \\
&= \frac{4}{(n\pi)^3} [1 - (-1)^n].
\end{aligned}$$

That is, we've found

$$B_n = \begin{cases} \frac{8}{(n\pi)^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

This suggests: **if** $\phi(x) = x(1 - x)$ can be expanded as in (18), then the expansion must be

$$S(x) = \frac{8}{\pi^3} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi x)}{(2n+1)^3}$$

That is a big **if**, however: is it actually true that $\phi(x) = S(x)$?

See [this](#) Desmos demo. Shown are $\phi(x)$, the N th partial sum $S_N(x)$, and $50 \cdot (\phi(x) - S_N(x))$. □

This example is suggestive, but when do we **know** that $\phi(x) = S(x)$?

Before establishing this, we consider a slightly different IBVP:

Example: We model a laterally insulated rod of length 1, perfectly insulated at the ends.

$$\begin{aligned}
u_t &= \alpha^2 u_{xx}, \\
u_x(0, t) &= 0, \\
u_x(1, t) &= 0, \quad 0 \leq t < \infty, \\
u(x, 0) &= \phi(x), \quad 0 \leq x \leq 1.
\end{aligned} \tag{20}$$

You will show in HW2 that all separable solutions to the BVP are

$$u_n(x, t) = A_n \cos(n\pi x) e^{-(n\pi\alpha)^2 t}, \quad n = 0, 1, 2, \dots$$

□

It is also an interesting question, then, whether an arbitrary function $\phi(x)$ on $[0, 1]$ may be written

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x). \tag{21}$$

As before, one may use the orthogonality of the function family $\{\cos(n\pi x)\}$,

$$\int_0^1 \cos(n\pi x) \cos(m\pi x) dx = \frac{1}{2} (1 + \delta_{n,0}) \delta_{n,m} = \begin{cases} \frac{1}{2} & n = m \neq 0 \\ 1 & n = m = 0 \\ 0 & n \neq m, \end{cases}$$

to deduce that, if an expansion (21) exists, the coefficients must be given by

$$A_n = 2 \int_0^1 \phi(x) \cos(n\pi x) dx \quad (22)$$

The series

$$C(x) := \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

with coefficients (22) is called the Fourier cosine series for $\phi(x)$ on $0 \leq x \leq 1$.

Day 6: Fourier Series (Lesson 11)

Do Fourier Series converge to the original function?

Both $S(x)$ and $C(x)$ arise as special cases of the more general Fourier series of a *periodic* function.

Given a periodic function $f(x)$ with period $2L$, one seeks to represent it as a combination of sines and cosines with compatible periods, i.e. drawn from the set

$$\{\sin(n\pi x/L), \cos(n\pi x/L)\}.$$

Such a representation would generally have the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

Similarly to before, one can straightforwardly show the orthogonality relations ($n, m \geq 0$)

$$\begin{aligned} \int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx &= L(1 - \delta_{n,0})\delta_{n,m}, \\ \int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx &= L(1 + \delta_{n,0})\delta_{n,m}, \\ \int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx &= 0, \end{aligned}$$

and these imply that, if an expansion as above exists, then the coefficients a_n and b_n must be

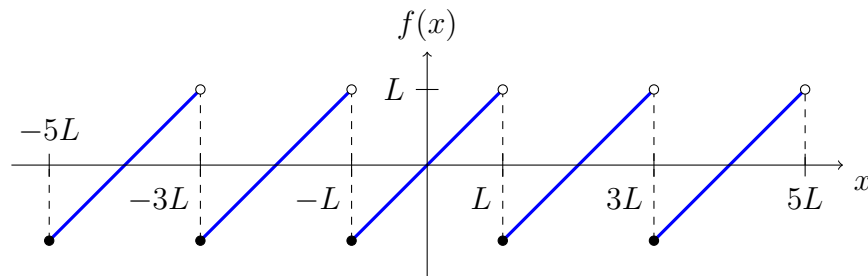
$$\boxed{\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}} \quad (23)$$

The series

$$\boxed{F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]} \quad (24)$$

with these a_n and b_n is called the Fourier series for $f(x)$.

Example: Consider the periodic function given by $f(x) = x$ on $[-L, L)$. This is the sawtooth wave:



We compute

$$a_n = \frac{1}{L} \int_{-L}^L x \cos\left(\frac{n\pi x}{L}\right) dx = 0$$

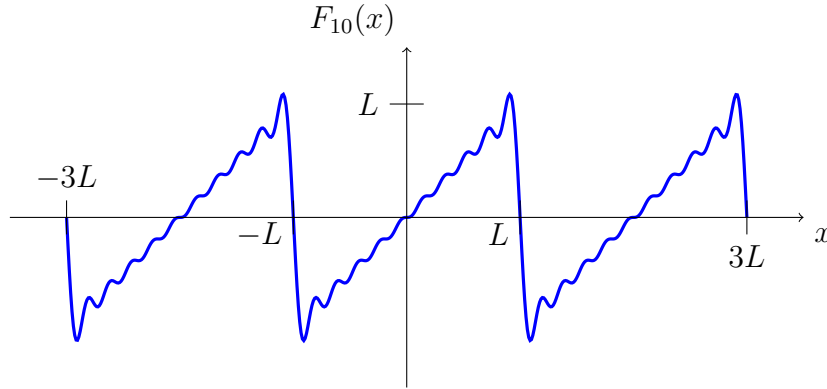
(how was this done immediately?), and

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^L x \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L} \cdot \frac{L}{n\pi} \left[-x \cos\left(\frac{n\pi x}{L}\right) \Big|_0^L + \int_0^L \cos\left(\frac{n\pi x}{L}\right) dx \right] \\ &= \frac{2L}{n\pi} (-1)^{n+1} \end{aligned}$$

Hence one has

$$\begin{aligned} F(x) &= \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{2L}{\pi} \left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2} \sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3} \sin\left(\frac{3\pi x}{L}\right) - \cdots \right] \end{aligned}$$

We may plot the partial sums $F_N(x)$ (see [this](#) Desmos plot with $L = 1$):



Such plots are suggestive, but is it true that $F(x) = f(x)$? A few values:

$$\begin{aligned} F(0) &= \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(0) = 0 = f(0) \quad \blacksquare \\ F(L/2) &= \frac{2L}{\pi} \left[\sin\left(\frac{\pi}{2}\right) - \frac{1}{2} \sin(\pi) + \frac{1}{3} \sin\left(\frac{3\pi}{2}\right) - \cdots \right] = \frac{2L}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \\ &= \frac{2L}{\pi} \tan^{-1}(1) = \frac{2L}{\pi} \cdot \frac{\pi}{4} = \frac{L}{2} = f(L/2) \quad \blacksquare \\ F(L) &= \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi) = 0 \neq -L = f(L) \quad \blacksquare \end{aligned}$$

This last result is perhaps not so surprising, as $x = L$ is a point of discontinuity for $f(x)$. □

Note: however many terms one includes in $F(x)$ for $\phi(x) = x$, the partial sums $F_N(x)$ always sharply oscillate and “overshoot” $\phi(x)$ near $x \rightarrow L$. This is known as the Gibbs phenomenon.

The Gibbs phenomenon is related to the failure of the Fourier series to converge *uniformly*.

The most intuitively natural notion of convergence is pointwise convergence, meaning

$$\lim_{N \rightarrow \infty} F_N(x_0) = f(x_0) \quad \Longleftrightarrow \quad \lim_{N \rightarrow \infty} |F_N(x_0) - f(x_0)| = 0$$

at each x_0 . Meanwhile, uniform convergence means

$$\lim_{N \rightarrow \infty} \max_x (|F_N(x) - f(x)|) = 0.$$

Both precisely when $F(x) = f(x)$ and the strength of the convergence has been widely studied.

There are many results establishing different notions of convergence under varying conditions.

The following result establishes pointwise convergence under suitable conditions:

Theorem (Fourier Convergence): If a periodic, (locally) integrable function f is differentiable at a point x_0 , then

$$F(x_0) = f(x_0).$$

Moreover, if f has a jump discontinuity at x_0 with left- and right-hand limits

$$\lim_{x \rightarrow x_0^-} f(x) = l, \quad \lim_{x \rightarrow x_0^+} f(x) = r$$

and the left- and right- hand derivatives of f exist at x_0 , then

$$F(x_0) = \frac{l + r}{2}$$

□

In the example, each discontinuity ($x_0 = kL$) has $l = L$, $r = -L$, and $F(x_0) = (l + r)/2 = 0$.

On the convergence rate of Fourier series, one has:

Theorem: If a periodic function $f(x)$ is piecewise smooth with jump discontinuities first appearing in the k th derivative, then

$$|a_n|, |b_n| < \frac{C}{n^{k+1}}$$

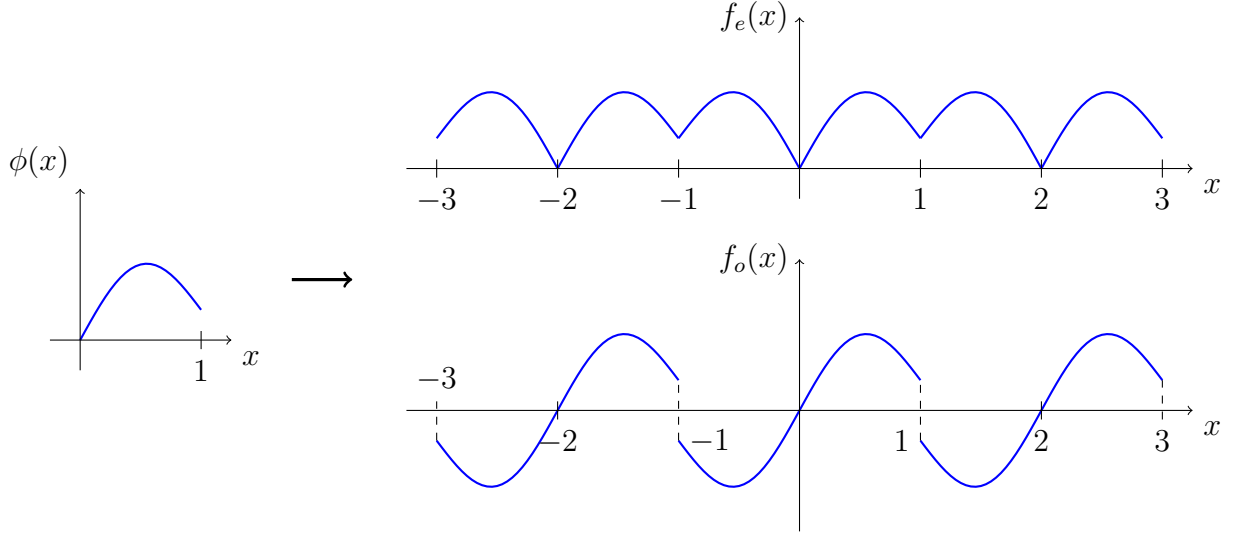
□

The smoother $f(x)$ is, then, the faster its Fourier series converges.

In the example, $f(x)$ itself was piecewise discontinuous ($k = 0$), and indeed $b_n \sim 1/n$.

There, $F(x)$ reduced to $S(x)$ because $f(x)$ was *odd*. $F(x)$ similarly reduces to $C(x)$ if $f(x)$ is *even*.

Any $\phi(x)$ on $[0, 1]$ in (14), (20) can extend to an odd $f_o(x)$ or even $f_e(x)$, periodic with period 2.



$C(x)$, $S(x)$ are built from $\phi(x)$. For the even $f_e(x)$: $F_e(x) = C(x)$. For the odd $f_o(x)$: $F_o(x) = S(x)$.

In this way, questions about $S(x)$ and $C(x)$ on $[0, 1]$ reduce to questions about $F(x)$ on $[-1, 1]$.

In particular, the theorems apply to the series $S(x)$ and $C(x)$ built to solve IBVP's.

Hence, we can solve (14) and (20) with effectively *any* reasonable $\phi(x)$ via separation of variables.

Example: We can construct the solution to the IBVP

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \\ u(0, t) &= 0, \\ u(1, t) &= 0, \quad 0 \leq t < \infty, \\ u(x, 0) &= x, \quad 0 \leq x \leq 1, \end{aligned}$$

by expanding $\phi(x) = x$ in its Fourier sine series on $[0, 1]$, as the separable solutions to the BVP are

$$u_n(x, t) = B_n \sin(n\pi x) e^{-(n\pi\alpha)^2 t}.$$

From the last example,

$$\phi(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x),$$

so the solution to the IBVP is simply

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x) e^{-(n\pi\alpha)^2 t}$$

To visualize, hit play on the s slider in [this](#) Desmos demo.

□

Day 7: Sturm-Liouville Theory (Lesson 7)

How can we solve more general homogeneous IBVP's?

Fourier series are powerful and arise in the canonical IBVP's, but they are not universal.

Example Consider a laterally insulated rod with temperature fixed to 0 on one end and submerged in water at temperature 0 on the other. The appropriate IBVP is

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \\ u(0, t) &= 0, \\ u_x(1, t) + hu(1, t) &= 0, \quad 0 \leq t < \infty, \\ u(x, 0) &= \phi(x), \quad 0 \leq x \leq 1, \end{aligned}$$

Since we have not modified our PDE, separation of variables plays out largely similar to before:

$$u(x, t) = X(x)T(t) \implies X(x)T'(t) = \alpha^2 X''(x)T(t) \implies \frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2,$$

giving the ODE $T' + (\alpha\lambda)^2 T = 0$ and the BVP

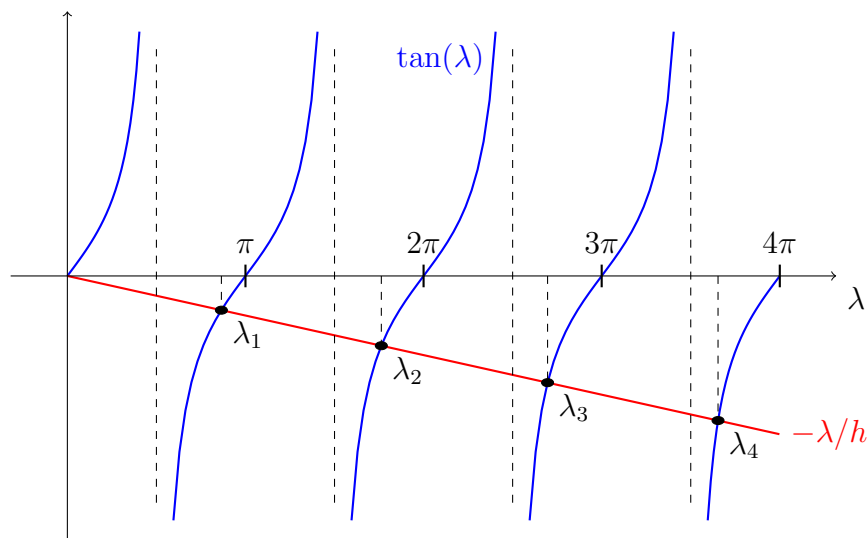
$$X'' + \lambda^2 X = 0, \quad X(0) = 0, \quad X'(1) + hX(1) = 0.$$

Hence $X(x) = A \cos(\lambda x) + B \sin(\lambda x)$, and the BC's impose

$$0 = X(0) = A, \quad 0 = X'(1) + hX(1) = B(\lambda \cos(\lambda) + h \sin(\lambda)),$$

so that nontrivial solutions to the BVP only arise for those λ values satisfying

$$\lambda \cos(\lambda) + h \sin(\lambda) = 0 \iff \tan(\lambda) = -\lambda/h.$$



There is an infinite sequence λ_n of solutions, $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ with $\lambda_n \rightarrow \infty$, and associated

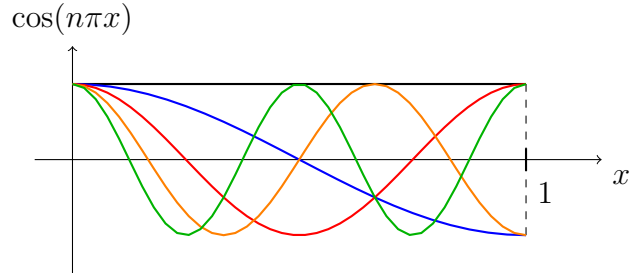
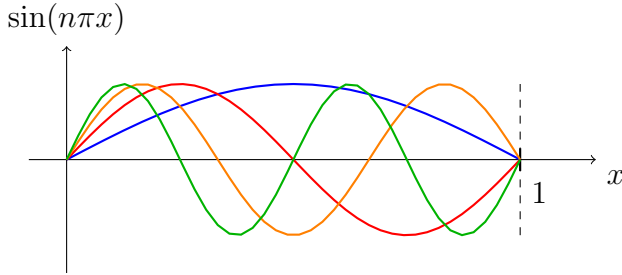
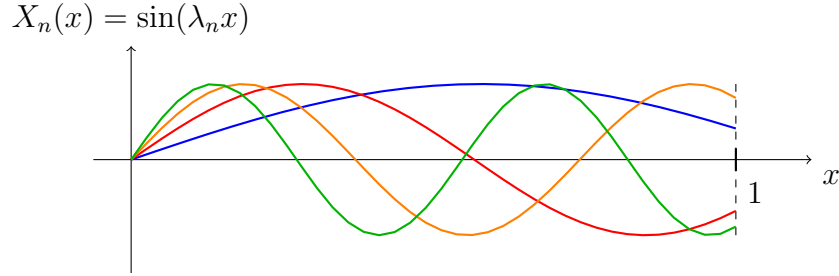
$$X_n(x) = \sin(\lambda_n x).$$

While we cannot write down a formula for λ_n , we have an infinite family of separable solutions

$$u_n(x, t) = C_n \sin(\lambda_n x) e^{-(\lambda_n \alpha)^2 t}$$

like before, out of which the IBVP solution could be built if one could expand

$$\phi(x) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n x).$$



Sines are involved, but this is distinctly not the Fourier sine series, nor any kind of Fourier series.

Even so, the family $\{\sin(\lambda_n x)\}_{n>0}$ does enjoy a (slightly less trivial) orthogonality property:

$$\int_0^1 \sin(\lambda_n x) \sin(\lambda_m x) dx = \left(\frac{1}{2} - \frac{\sin(2\lambda_n)}{4\lambda_n} \right) \delta_{n,m},$$

which can be used deduce that the above coefficients C_n would have to be.

$$C_n = \frac{4\lambda_n}{2\lambda_n - \sin(2\lambda_n)} \int_0^1 \phi(x) \sin(\lambda_n x) dx.$$

One can see (approximate) convergence to $\phi(x) = x$ and evolution of $u(x, t)$ in [this](#) Desmos demo.

While this problem was not reducible to Fourier series, a very similar pattern emerged.

Did we get lucky? How broad are these patterns?

□

When one solves a spatially 1D heat-type (see HW3) PDE + BC's combo (each linear and homogeneous) on $a \leq x \leq b$ via separation of variables, one obtains a BVP for $X(x)$ of the form

$$\begin{aligned} (p(x)y')' - q(x)y + \lambda w(x)y &= 0 \\ \alpha_1 y(a) + \beta_1 y'(a) &= 0 \\ \alpha_2 y(b) + \beta_2 y'(b) &= 0 \end{aligned} \tag{25}$$

This is a Sturm-Liouville BVP: regular if $p(x), w(x) > 0$ on $a \leq x \leq b$ and p, p', q, w are continuous.

Defining the linear differential operator L by

$$L[y] = \frac{1}{w(x)} [-(p(x)y')' + q(x)y],$$

we see that the Sturm-Liouville ODE is equivalent to

$$L[y] = \lambda y.$$

This is an eigenvalue problem, posed on the space of functions on $[a, b]$ satisfying the BC's.

Nontrivial solutions $y(x)$ are called eigenfunctions, and the associated λ is called an eigenvalue.

Example: The prototypical Sturm-Liouville problems are those we've seen in our IBVP's leading to Fourier series:

$$\begin{aligned} X'' + \lambda X &= 0, & X(0) &= 0, & X(1) &= 0, \\ Y'' + \lambda Y &= 0, & Y'(0) &= 0, & Y'(1) &= 0. \end{aligned}$$

These had $p(x) = w(x) = 1$ and $q(x) = 0$. They admit an infinite sequence of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty$$

(given by $(n\pi)^2$) with eigenfunctions

$$X_n(x) = \sin(n\pi x), \quad Y_n(x) = \cos(n\pi x).$$

These families of eigenfunctions each enjoyed orthogonality relations,

$$\int_0^1 X_n(x)X_m(x)dx \propto \delta_{n,m}, \quad \int_0^1 Y_n(x)Y_m(x)dx \propto \delta_{n,m},$$

and any “nice” (see the theorem on Day 6) function $\phi(x)$ on $(0, 1)$ could be expanded

$$\phi(x) = \sum_n B_n X_n(x) = \sum_n A_n Y_n(x).$$

□

Outside of explicit formulas, all of these properties generalize to any regular Sturm-Liouville BVP.

On the space of real-valued functions on $[a, b]$ satisfying the BC's in (25), define the inner product

$$\langle f, g \rangle := \int_a^b f(x)g(x)w(x)dx.$$

We say that f and g are orthogonal (with respect to the weight function $w(x)$) if $\langle f, g \rangle = 0$.

We demonstrate an essential property of the differential operator L on this space:

$$\begin{aligned} \langle f, L[g] \rangle - \langle L[f], g \rangle &= \int_a^b f(x) [-(p(x)g'(x))' + q(x)g(x)] dx \\ &\quad - \int_a^b g(x) [-(p(x)f'(x))' + q(x)f(x)] dx \end{aligned}$$

$$\begin{aligned}
&= \int_a^b [g(x)(p(x)f'(x))' - f(x)(p(x)g'(x))'] dx \\
&= \int_a^b \frac{d}{dx} [p(x)(f'(x)g(x) - f(x)g'(x))] dx \\
&= p(x)(f'(x)g(x) - f(x)g'(x)) \Big|_a^b \\
&= 0
\end{aligned}$$

One should check that the last step follows from f and g satisfying the BC's in (25). We've found

$$\langle f, L[g] \rangle = \langle L[f], g \rangle,$$

and L is therefore said to be self-adjoint. With this, we may readily conclude

Theorem: Eigenfunctions with distinct eigenvalues are orthogonal.

Proof. If eigenfunctions ϕ_1, ϕ_2 of (25) have eigenvalues $\lambda_1 \neq \lambda_2$, then

$$0 = \langle \phi_1, L[\phi_2] \rangle - \langle L[\phi_1], \phi_2 \rangle = \langle \phi_1, \lambda_2 \phi_2 \rangle - \langle \lambda_1 \phi_1, \phi_2 \rangle = (\lambda_2 - \lambda_1) \langle \phi_1, \phi_2 \rangle.$$

Since $\lambda_2 - \lambda_1 \neq 0$, we must have $\langle \phi_1, \phi_2 \rangle = 0$. □

In the computation leading to self-adjointness above, we found

$$w(x)(fL[g] - L[f]g) = \frac{d}{dx} [p(x)(f'(x)g(x) - f(x)g'(x))]$$

This can also be leveraged to deduce that an eigenvalue has only one independent eigenfunction:

Theorem Eigenfunctions with the same eigenvalue are multiples of each other.

Proof. If ϕ_1 and ϕ_2 have the same eigenvalue λ , then

$$\frac{d}{dx} [p(x)(\phi_1'(x)\phi_2(x) - \phi_1(x)\phi_2'(x))] = w(x)(\phi_1 L[\phi_2] - L[\phi_1]\phi_2) = w(x)(\lambda\phi_1\phi_2 - \lambda\phi_1\phi_2) = 0$$

That is, the quantity

$$p(x)[\phi_1'\phi_2 - \phi_1\phi_2']$$

is constant. The BC's impose that this is zero at a and b , so it is zero everywhere.

Recall from ODE's that $W(x) := \phi_1'\phi_2 - \phi_1\phi_2'$ is the Wronskian of ϕ_1 and ϕ_2 , and $W(x) \equiv 0$ implies that ϕ_1 is a constant multiple of ϕ_2 . □

The analogue to Fourier series is beginning to emerge. It is completed by the following result, whose proof is (well) beyond this course.

Theorem: A regular Sturm-Liouville BVP (25) admits an infinite sequence of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty$$

with associated eigenfunctions ϕ_n . Any continuously differentiable $\phi(x)$ on $[a, b]$ may be expanded

$$\phi(x) = \sum_{n=1}^{\infty} C_n \phi_n(x),$$

converging pointwise on $a < x < b$, where C_n is given (from the orthogonality relation) by

$$C_n = \frac{\langle \phi, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_a^b \phi(x) \phi_n(x) w(x) dx}{\int_a^b \phi_n^2(x) w(x) dx}$$

If $\phi(x)$ also satisfies the BC's, the series is also guaranteed to converge at $x = a, b$.

□

Later, we'll see irregular SL BVP's, wherein these properties are not automatically guaranteed.

Day 8: Inhomogeneous IBVP's (Lessons 6)

How can we handle inhomogeneous problems?

Our strategies so far have hinged on linear combinations of BVP solutions remaining solutions, and this required the BVP to be homogeneous. Inhomogeneous problems require further ingenuity.

First: to what extent can inhomogeneous problems be reduced to homogeneous ones?

Would like to draw on the superposition principle like in ODE's, where one often uses the schematic

$$(\text{general solution}) = (\text{homogeneous solution}) + (\text{particular solution}). \quad (26)$$

Somewhat complicated by having two sources of inhomogeneity: both the PDE and the BC's.

Example: Consider a laterally insulated rod whose two ends are kept at temperatures k_1, k_2 :

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \\ u(0, t) &= k_1, \\ u(1, t) &= k_2, \quad 0 \leq t < \infty, \\ u(x, 0) &= \phi(x), \quad 0 \leq x \leq 1. \end{aligned}$$

Separable solutions exist, but are not be useful because linear combinations do not respect the BC's.

We could solve this with homogeneous BC's $u(0, t) = u(1, t) = 0$, so we seek a decomposition (26).

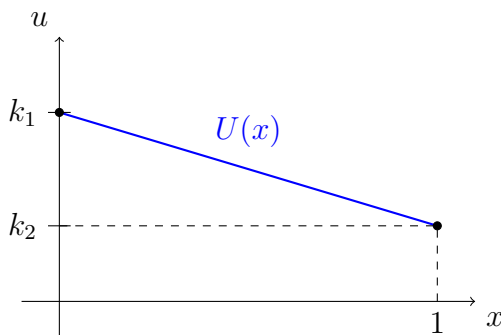
How can we construct a particular solution? If it exists, the *steady-state* is an ideal candidate.

Recall from HW2: the steady-state is a BVP solution $u(x, t) = U(x)$. Here, $U(x)$ solves the BVP

$$U''(x) = 0, \quad U(0) = k_1, \quad U(1) = k_2,$$

so the graph of $U(x)$ must be the line connecting $(0, k_1)$ and $(1, k_2)$ in the xu plane,

$$U(x) = k_1(1 - x) + k_2x. \quad (27)$$



Defining $w(x, t) := u(x, t) - U(x)$, we may easily compute

$$w_t = u_t, \quad w_{xx} = u_{xx} - U'' = u_{xx}, \quad w(0, t) = w(1, t) = 0, \quad w(x, 0) = \phi(x) - U(x),$$

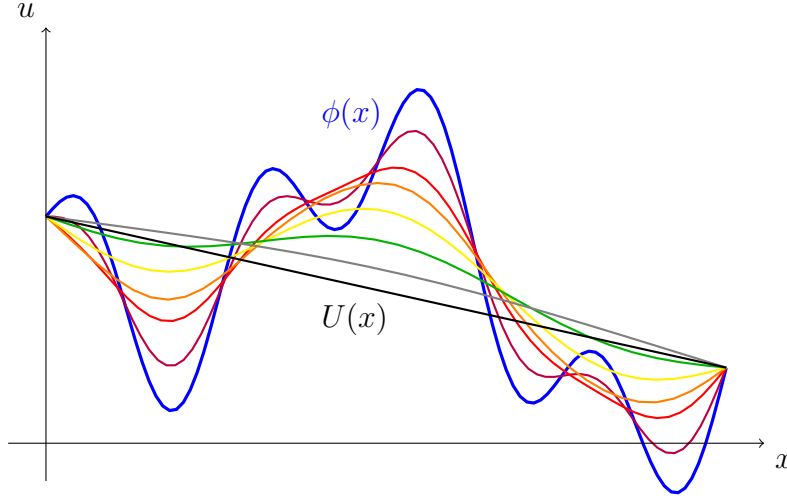
so w solves the IBVP

$$\begin{aligned} v_t &= \alpha^2 v_{xx}, \\ v(0, t) &= 0, \\ v(1, t) &= 0, \quad 0 \leq t < \infty, \\ v(x, 0) &= \phi(x) - U(x), \quad 0 \leq x \leq 1. \end{aligned}$$

As this is homogeneous, we may readily solve by expanding $\phi(x) - U(x)$ in a Fourier sine series.

Once $v(x, t)$ is known, we simply add back the steady-state (27), $u(x, t) = v(x, t) + U(x)$.

Snapshots of a typical solution are shown below. See also [this](#) Desmos demo.



□

The above only works if sources and BC's have no explicit time dependence, so a steady-state exists.

Generally, we cannot entirely eliminate inhomogeneity. Consider the IBVP with general linear BC's:

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \\ \alpha_1 u(0, t) + \beta_1 u_x(0, t) &= g_1(t), \\ \alpha_2 u(1, t) + \beta_2 u_x(1, t) &= g_2(t), \quad 0 \leq t < \infty, \\ u(x, 0) &= \phi(x), \quad 0 \leq x \leq 1. \end{aligned}$$

Given the time-dependent BC's, a fully steady-state solution will not exist.

We may still take inspiration from (27), however, and subtract off an analogous component,

$$U(x, t) = A(t)(1 - x) + B(t)x,$$

with $A(t)$ and $B(t)$ chosen so that $U(x, t)$ solves the BC's. This imposes the constraints

$$\begin{aligned} g_1(t) &= \alpha_1 A(t) + \beta_1 (B(t) - A(t)), \\ g_2(t) &= \alpha_2 B(t) + \beta_2 (B(t) - A(t)), \end{aligned}$$

which may be straightforwardly solved for $A(t)$, $B(t)$ given (almost) any values of α_i , β_i .

Knowing $A(t)$ and $B(t)$, we define as before $v(x, t) := u(x, t) - U(x, t)$, which now solves the IBVP

$$\begin{aligned} v_t &= \alpha^2 v_{xx} - U_t, \\ \alpha_1 v(0, t) + \beta_1 v_x(0, t) &= 0, \\ \alpha_2 v(1, t) + \beta_2 v_x(1, t) &= 0, & 0 \leq t < \infty, \\ v(x, 0) &= \phi(x) - U(x, 0), & 0 \leq x \leq 1. \end{aligned}$$

We have managed to homogenize the BC 's, but destroyed the homogeneity of the PDE .

Example: The natural case $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 0$ in the above simply yields

$$A(t) = g_1(t), \quad B(t) = g_2(t),$$

and the IBVP for v becomes

$$\begin{aligned} v_t &= \alpha^2 v_{xx} - g_1'(t)(1-x) - g_2'(t)x, \\ v(0, t) &= 0, \\ v(1, t) &= 0, & 0 \leq t < \infty, \\ v(x, 0) &= \phi(x) - g_1(0)(1-x) + g_2(0)x, & 0 \leq x \leq 1. \end{aligned}$$

□

You will treat the case $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 1$ in HW4.

The moral: we generally cannot arrange for homogeneity of *both* the PDE and BC's, but we can always at least ensure *one* is homogeneous. We will prefer homogeneous BC's.

Day 9: Inhomogeneous IBVP's: Eigenfunction Expansion (Lesson 9)

How can we construct solutions to inhomogeneous IBVP's?

Consider the broad class of heat-type problems with homogeneous BC's:

$$\begin{aligned} u_t &= \frac{1}{w(x)} [(p(x)u_x)_x - q(x)u] + f(x, t), \\ \alpha_1 u(0, t) + \beta_1 u_x(0, t) &= 0, \\ \alpha_2 u(1, t) + \beta_2 u_x(1, t) &= 0, & 0 \leq t < \infty, \\ u(x, 0) &= \phi(x), & 0 \leq x \leq 1. \end{aligned} \quad (28)$$

We develop the eigenfunction expansion method, similar to *variation of parameters* in ODE's.

As in ODE's, the essential first step is to consider the fully homogeneous case, $f \rightarrow 0$.

In that case, HW3 shows: separating $u(x, t) = X(x)T(t)$ yields the standard SL BVP (25) for $X(x)$.

The homogeneous BVP then admits separable solutions $u_n(x, t) = \phi_n(x)e^{-\lambda_n t}$, and more generally

$$u_h(x, t) = \sum_n C_n \phi_n(x) e^{-\lambda_n t}. \quad (29)$$

Returning to the full problem (28), SL theory ensures we may expand $f(x, t)$ in the ϕ_n at each t :

$$f(x, t) = \sum_n f_n(t) \phi_n(x)$$

The eigenfunction expansion method takes as an ansatz the form (29), but promoting $C_n \rightarrow C_n(t)$,

$$u(x, t) = \sum_n C_n(t) \phi_n(x) e^{-\lambda_n t}$$

Substituting this into the PDE (and using that the ϕ_n are SL BVP eigenfunctions) yields

$$\begin{aligned} \sum_n C'_n(t) \phi_n(x) e^{-\lambda_n t} &= \sum_n f_n(t) \phi_n(x) \\ \implies \sum_n (C'_n(t) e^{-\lambda_n t} - f_n(t)) \phi_n(x) &= 0 \\ \implies C'_n(t) e^{-\lambda_n t} - f_n(t) &= 0, \end{aligned}$$

by orthogonality of $\{\phi_n\}$. That is, we've found

$$C_n(t) = \int f_n(t) e^{\lambda_n t} dt$$

Schematically, our ansatz reduces the PDE to an infinitum of ODE's, each characterizing the response to the source component $f_n(t)\phi_n(x)$. Finally, we impose the IC to fully determine $C_n(t)$:

$$\phi(x) = u(x, 0) = \sum_n C_n(0) \phi_n(x).$$

The initial values $C_n(0)$, then, are precisely the coefficients in the eigenfunction expansion of $\phi(x)$,

$$C_n(0) = \frac{\langle \phi, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_a^b \phi(x) \phi_n(x) w(x) dx}{\int_a^b \phi_n^2(x) w(x) dx}.$$

Combining our results, the coefficients $C_n(t)$ are fully given by

$$C_n(t) = C_n(0) + \int_0^t f_n(s) e^{\lambda_n s} ds$$

Example: To illustrate, consider the simple IBVP (in which one could shift away the steady-state)

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + \sin(3\pi x), \\ u(0, t) &= 0, \\ u(1, t) &= 0, & 0 \leq t < \infty, \\ u(x, 0) &= \sin(\pi x), & 0 \leq x \leq 1. \end{aligned}$$

The BVP eigenfunctions are $\phi_n(x) = \sin(n\pi x)$ for $n > 0$, and in the present notation, $\lambda_n = (n\pi\alpha)^2$.

Hence $f(x, t) = \sin(3\pi x)$ simply has $f_n(t) = \delta_{n,3}$ and $\phi(x) = \sin(\pi x)$ gives $C_n(0) = \delta_{n,1}$, so that

$$C_n(t) = C_n(0) + \int_0^t f_n(s) e^{\lambda_n s} ds = \delta_{n,1} + \delta_{n,3} \int_0^t e^{\lambda_n s} ds = \delta_{n,1} + \delta_{n,3} \frac{e^{\lambda_n t} - 1}{\lambda_n}.$$

Namely, we have $C_n(t) = 0$ for all n except

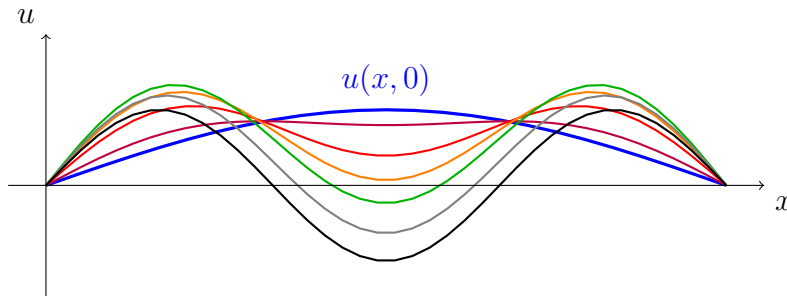
$$C_1(t) = 1, \quad C_3(t) = \frac{e^{\lambda_3 t} - 1}{\lambda_3},$$

and hence our solution $u(x, t)$ is

$$\begin{aligned} u(x, t) &= \phi_1(x) e^{-\lambda_1 t} + \frac{e^{\lambda_3 t} - 1}{\lambda_3} \phi_3(x) e^{-\lambda_3 t} \\ &= \phi_1(x) e^{-\lambda_1 t} + \frac{1}{\lambda_3} \phi_3(x) (1 - e^{-\lambda_3 t}) \end{aligned}$$

$$u(x, t) = e^{-(\pi\alpha)^2 t} \sin(\pi x) + \frac{1}{(3\pi\alpha)^2} \left[1 - e^{-(3\pi\alpha)^2 t} \right] \sin(3\pi x)$$

Some snapshots of this evolution with $3\pi\alpha = 1$ are shown below. See also [Desmos](#).



□

To more completely illustrate, let us work a more involved example.

Example: Beginning at $t = 0$, dye is injected at $x = L/2$ (cm) at a rate of $r \frac{\text{g}}{\text{s}}$ for 1 s into a very thin (impermeable) tube of initially pure water that flows to the right at rate $v \frac{\text{cm}}{\text{s}}$. Filters at $x = 0$ and $x = L$ remove all dye from the water at these positions. What is the linear concentration $\rho(x, t)$ ($\frac{\text{g}}{\text{cm}}$) of the dye?

Recalling Day 3 and denoting by D the diffusion coefficient ($\frac{\text{cm}^2}{\text{s}}$), the appropriate IBVP is

$$\begin{aligned} \rho_t &= D\rho_{xx} - v\rho_x + rH(1-t)\delta(x - L/2), \\ \rho(0, t) &= 0, \\ \rho(L, t) &= 0, & 0 \leq t < \infty, \\ \rho(x, 0) &= 0, & 0 \leq x \leq L, \end{aligned} \tag{30}$$

where $H(t)$ is the *Heaviside step function* and $\delta(x)$ is the *Dirac delta “function”*.

We find separable homogeneous solutions. Plugging $\rho(x, t) = X(x)T(t)$ into the sourceless PDE:

$$T'(t)X(x) = \rho_t = D\rho_{xx} - v\rho_x = (DX''(x) - vX'(x))T(t),$$

so that

$$\frac{T'(t)}{T(t)} = \frac{DX''(x) - vX'(x)}{X(x)} = -\lambda,$$

so $T' = -\lambda T$ and X satisfies the BVP

$$DX'' - vX' + \lambda X = 0, \quad X(0) = X(L) = 0.$$

Multiplying by $e^{-vx/D}$, we see this is a SL BVP with $p(x) = Dw(x) = De^{-vx/D}$:

$$(De^{-vx/D}X')' + \lambda e^{-vx/D}X = 0, \quad X(0) = X(L) = 0.$$

Defining $\mu := \sqrt{\frac{\lambda}{D} - (\frac{v}{2D})^2}$, it is a straightforward ODE exercise to see that solutions are

$$X(x) = Be^{vx/(2D)} \sin(\mu x)$$

for μ such that $\sin(\mu L) = 0$, i.e. $\mu_n = \frac{n\pi}{L}$. That is, our eigenvalues are

$$\lambda_n = \left(\mu_n^2 + \left(\frac{v}{2D} \right)^2 \right) D = \frac{D}{L^2} (n\pi)^2 + \frac{v^2}{4D} = \frac{D}{L^2} \left((n\pi)^2 + \left(\frac{Lv}{2D} \right)^2 \right)$$

with eigenfunctions

$$\phi_n(x) = e^{vx/(2D)} \sin\left(\frac{n\pi x}{L}\right).$$

Returning to the full problem (30), we presume a solution of the form

$$\rho(x, t) = \sum_n C_n(t) \phi_n(x) e^{-\lambda_n t}.$$

Our IC is trivial, but we must expand the source $f(x, t) = rH(1 - t)\delta(x - L/2)$

$$\begin{aligned} f(x, t) &= \sum_n f_n(t) \phi_n(x) = e^{vx/(2D)} \sum_n f_n(t) \sin\left(\frac{n\pi x}{L}\right) \\ \iff f(x, t)e^{-vx/(2D)} &= \sum_n f_n(t) \sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

Either observing this is the Fourier sine series of $f(x, t)e^{-vx/(2D)}$ (on $0 \leq x \leq L$) or using the weighted inner product $\langle \cdot, \cdot \rangle$, we see

$$\begin{aligned} f_n(t) &= \frac{2}{L} \int_0^L f(x, t) e^{-vx/(2D)} \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2r}{L} H(1 - t) \int_0^L \delta(x - L/2) e^{-vx/(2D)} \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2r}{L} e^{-\frac{vL}{4D}} \sin\left(\frac{n\pi}{2}\right) H(1 - t), \end{aligned}$$

and hence the coefficients $C_n(t)$ are

$$\begin{aligned} C_n(t) &= \int_0^t f_n(s) e^{\lambda_n s} ds = \frac{2r}{L} e^{-vL/(4D)} \sin\left(\frac{n\pi}{2}\right) \int_0^t H(1 - s) e^{\lambda_n s} ds \\ &= \frac{2r}{L} e^{-\frac{vL}{4D}} \sin\left(\frac{n\pi}{2}\right) \int_0^{\min(t, 1)} e^{\lambda_n s} ds \\ &= \frac{2r}{L\lambda_n} \sin\left(\frac{n\pi}{2}\right) e^{-\frac{vL}{4D}} (e^{\min(t, 1)\lambda_n} - 1), \end{aligned}$$

and the solution $\rho(x, t)$ becomes

$$\begin{aligned} \rho(x, t) &= \frac{2r}{L} e^{\frac{v(2x-L)}{4D}} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{\lambda_n} \sin\left(\frac{n\pi x}{L}\right) (e^{\min(t, 1)\lambda_n} - 1) e^{-\lambda_n t} \\ &= \boxed{\frac{2r}{L} e^{\frac{Lv}{4D}(\frac{2x}{L}-1)} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{\lambda_n} \sin\left(\frac{n\pi x}{L}\right) (e^{\min(0, 1-t)\lambda_n} - e^{-\lambda_n t})} \end{aligned}$$

While a bit complicated, this can be evaluated however accurately one likes given D, L, v , and r .

Some relevant length scales are L and $\frac{D}{v}$; some relevant time scales are 1 s, $\frac{L}{v}$, $\frac{D}{v^2}$, and $\frac{L^2}{D}$.

To understand this solution, adjust sliders in [this](#) Desmos demo (with $L = 100$). Some observations:

- The rate r only affects the overall amplitude of ρ .
- The dimensionless quantity $\frac{Lv}{D}$ measures the ability of diffusion to overcome convection and strongly impacts the convergence rate near $x \rightarrow L$.
- The terms decay only like $\sim \frac{1}{\lambda_n}$ for $0 < t \leq 1$, but like $\sim \frac{e^{-\lambda_n(t-1)}}{\lambda_n}$ for $t > 1$.
- The time scale $\frac{L^2}{D}$ determines how quickly convergence improves for $t > 1$.

□

Day 10: Fourier Transform (Lessons 11-12)

How can we solve IBVP's on infinite intervals?

We are now equipped, in principle, to solve any 1D heat-type IBVP on a finite interval $0 \leq x \leq L$.

What if the spatial domain is unbounded, say $-\infty < x < \infty$?

To facilitate this, we conceptually extend the Fourier series (24) of a periodic function $f(x)$,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right].$$

It is useful to reformat. Recalling *Euler's identity*

$$e^{ix} = \cos(x) + i \sin(x),$$

we may rewrite the n th term in the Fourier series

$$\begin{aligned} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) &= \frac{a_n}{2} [e^{in\pi x/L} + e^{-in\pi x/L}] + \frac{b_n}{2i} [e^{in\pi x/L} - e^{-in\pi x/L}] \\ &= \frac{a_n - ib_n}{2} e^{in\pi x/L} + \frac{a_n + ib_n}{2} e^{-in\pi x/L}. \end{aligned}$$

Recalling the definitions (23) of a_n and b_n , we see that

$$\begin{aligned} \frac{a_n - ib_n}{2} &= \frac{1}{2L} \int_{-L}^L f(x) \left[\cos\left(\frac{n\pi x}{L}\right) - i \sin\left(\frac{n\pi x}{L}\right) \right] dx = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx, \\ \frac{a_n + ib_n}{2} &= \frac{1}{2L} \int_{-L}^L f(x) \left[\cos\left(\frac{n\pi x}{L}\right) + i \sin\left(\frac{n\pi x}{L}\right) \right] dx = \frac{1}{2L} \int_{-L}^L f(x) e^{in\pi x/L} dx. \end{aligned}$$

Denoting the complex numbers

$$\hat{f}_n := \frac{1}{\sqrt{2L}} \int_{-L}^L f(x) e^{-in\pi x/L} dx, \quad (31)$$

then, we've found

$$a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) = \frac{1}{\sqrt{2L}} \left[\hat{f}_n e^{in\pi x/L} + \hat{f}_{-n} e^{-in\pi x/L} \right],$$

and we may now much more succinctly express the Fourier series (24)

$$\boxed{F(x) = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} \hat{f}_n e^{in\pi x/L}} \quad (32)$$

$F(x) = f(x)$ says: the information of a *periodic function* $f(x)$ is equivalent to that of a *sequence* $(\hat{f}_n)$.

The magnitudes $|\hat{f}_n|$ measure the component of $f(x)$ with frequency $\frac{n\pi}{L}$: this is the spectrum of $f(x)$.

A periodic function has a discrete spectrum, or only a discrete collection of frequencies $\{\frac{n\pi}{L}\}$.

In the limit $L \rightarrow \infty$, the stepsize $\frac{\pi}{L}$ between frequencies approaches 0, suggesting that an arbitrary function $f(x)$ on $\mathbb{R}$ should be comprised of a *continuum* of frequencies.

Analogously to (31), then: if $f(x)$ is integrable on $\mathbb{R}$, we define the Fourier transform of $f(x)$ as

$$\mathcal{F}[f](\xi) = \hat{F}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx,$$

where ξ represents a continuous frequency variable, being the frequency of the oscillatory $e^{-i\xi x}$.

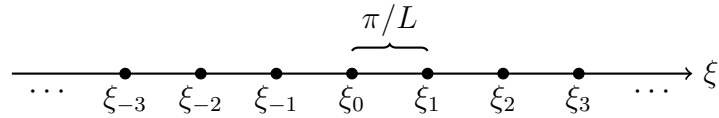
For very large L , one might approximate (31) as (denoting $\xi_n = \frac{n\pi}{L}$)

$$\hat{f}_n = \frac{1}{\sqrt{2L}} \int_{-L}^L f(x) e^{-i\xi_n x} dx \approx \sqrt{\frac{\pi}{L}} \hat{F}(\xi_n),$$

and hence (32) reads (for the usual “nice” $f(x)$)

$$f(x) = F(x) \approx \frac{\sqrt{\pi}}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} \hat{F}(\xi_n) e^{i\xi_n x} = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \left(\hat{F}(\xi_n) e^{i\xi_n x} \right) \cdot \frac{\pi}{L}$$

In this final expression, the function $\hat{F}(\xi) e^{i\xi x}$ is evaluated at the points $\xi = \xi_n$, spaced $\frac{\pi}{L}$ apart.



This is precisely a *Riemann sum* for the integral of $\hat{F}(\xi) e^{i\xi x}$ over $\xi \in \mathbb{R}$, suggesting (taking $L \rightarrow \infty$)

$$f(x) \stackrel{?}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(\xi) e^{i\xi x} d\xi$$

This reasoning is indeed only a suggestion. However, it can be proven under suitable conditions:

Theorem (Fourier Inversion Theorem): If $f(x)$ is continuous and both $f(x)$ and its transform $\hat{F}(\xi)$ are absolutely integrable over $\mathbb{R}$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(\xi) e^{i\xi x} d\xi$$

□

For this reason the inverse Fourier transform operator $\mathcal{F}^{-1}$ is defined as

$$\mathcal{F}^{-1}[\hat{F}](x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(\xi) e^{i\xi x} d\xi$$

The FIT, then, simply affirms $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$. The transforms $\mathcal{F}$, $\mathcal{F}^{-1}$ are linear operators.

For future reference, $\mathcal{F}$ and $\mathcal{F}^{-1}$ continue to be useful tools in higher dimensional settings.

For integrable functions $f(\vec{\mathbf{x}})$, $\widehat{F}(\vec{\xi})$ on $\mathbb{R}^n$, these operators are defined according to

$$\boxed{\mathcal{F}[f](\vec{\xi}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\vec{\mathbf{x}}) e^{-i\vec{\mathbf{x}} \cdot \vec{\xi}} d^n \vec{\mathbf{x}} \quad \mathcal{F}^{-1}[\widehat{F}](\vec{\mathbf{x}}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{F}(\vec{\xi}) e^{i\vec{\mathbf{x}} \cdot \vec{\xi}} d^n \vec{\xi}}$$

Day 11: Fourier Transforms in IVP's (Lesson 12)

How can we solve IVP's on infinite intervals?

We investigate structure and properties of Fourier transforms.

Example: We may readily compute the Fourier transform of $f(x) = e^{-ax^2}$:

$$\begin{aligned}\widehat{F}(\xi) &= \mathcal{F}[e^{-ax^2}](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2 - i\xi x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left[-a\left(x + \frac{i\xi}{2a}\right)^2 - \frac{\xi^2}{4a}\right]} dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(x + \frac{i\xi}{2a}\right)^2} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{4a}} \int_{-\infty}^{\infty} e^{-az^2} dz = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}},\end{aligned}$$

where we have changed variables¹¹ to $z = x + \frac{i\xi}{2a}$ and used the standard result

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}.$$

We may verify the FIT's conclusion for this example:

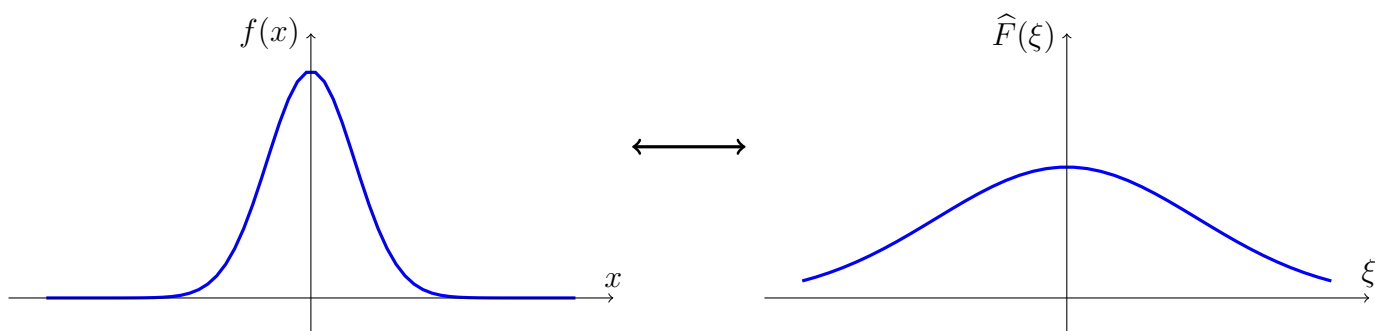
$$\mathcal{F}^{-1}[\widehat{F}(\xi)] = \frac{1}{\sqrt{2a}} \mathcal{F}^{-1}[e^{-\frac{\xi^2}{4a}}]$$

Invoking the same computation as above with $a \rightarrow b = \frac{1}{4a}$ (and a complex conjugation) gives

$$\mathcal{F}^{-1}[e^{-b\xi^2}](x) = \frac{1}{\sqrt{2b}} e^{-\frac{x^2}{4b}} = \sqrt{2a} e^{-ax^2},$$

so that we indeed have

$$\mathcal{F}^{-1}[\widehat{F}](x) = e^{-ax^2} = f(x).$$



□

How does this help us to solve IBVP's specified on $-\infty < x < \infty$?

This is provided by how the Fourier transform interacts with derivatives:

Theorem: If $f(x)$ and $f'(x)$ are absolutely integrable over $\mathbb{R}$ and $\lim_{x \rightarrow \pm\infty} f(x) = 0$, then

$$\mathcal{F}[f'](\xi) = i\xi \mathcal{F}[f](\xi).$$

¹¹There is some complex analysis sleight-of-hand here: the integral with respect to z should technically be over the complex contour from $-\infty + \frac{i\xi}{2a}$ to $\infty + \frac{i\xi}{2a}$. Take a course in complex analysis to learn why this is the same as the integral over $\mathbb{R}$ (for this integrand!).

Proof. We simply integrate by parts:

$$\mathcal{F}[f'](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\xi x} dx = \frac{1}{\sqrt{2\pi}} (f(x) e^{-i\xi x}) \Big|_{-\infty}^{\infty} + \frac{i\xi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx = i\xi \mathcal{F}[f](\xi)$$

□

Under appropriate hypotheses, this may be iterated to obtain

$$\mathcal{F}[f''](\xi) = i\xi \mathcal{F}[f'](\xi) = -\xi^2 \mathcal{F}[f](\xi).$$

That is, $\mathcal{F}$ turns differential operations in x into algebraic operations in ξ (the *frequency domain*).

Given a PDE for $u(x, t)$ on $-\infty < x < \infty$, we apply $\mathcal{F}$ (at each t) to obtain a problem in $\widehat{u}(\xi, t)$,

$$\widehat{u}(\xi, t) = \mathcal{F}[u] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-i\xi x} dx$$

We have the following¹² mappings under this application:

$$\begin{array}{ccc} & \mathcal{F} & \\ u & \longmapsto & \widehat{u} \\ u_x & \longmapsto & i\xi \widehat{u} \\ u_{xx} & \longmapsto & -\xi^2 \widehat{u} \\ u_t & \longmapsto & \widehat{u}_t \end{array}$$

To illustrate, consider the IVP¹³

$$u_t = u_{xx}, \quad u(x, 0) = \phi(x) \text{ for } x \in \mathbb{R}$$

Applying $\mathcal{F}$ to both the PDE and the IC yields

$$\widehat{u}_t = -\xi^2 \widehat{u}, \quad \widehat{u}(\xi, 0) = \widehat{\Phi}(\xi) \text{ for } \xi \in \mathbb{R}$$

Crucially: for each fixed ξ , the PDE has become an ODE in t that can be easily solved,

$$\widehat{u}(\xi, t) = \widehat{u}(\xi, 0) e^{-\xi^2 t} = \widehat{\Phi}(\xi) e^{-\xi^2 t}$$

We may now recover $u(x, t)$ by applying $\mathcal{F}^{-1}$,

$$\boxed{u(x, t) = \mathcal{F}^{-1}[\widehat{u}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\Phi}(\xi) e^{-\xi^2 t} e^{i\xi x} d\xi} \quad (33)$$

This may or may not be analytically evaluable, but it can always be approximated numerically.

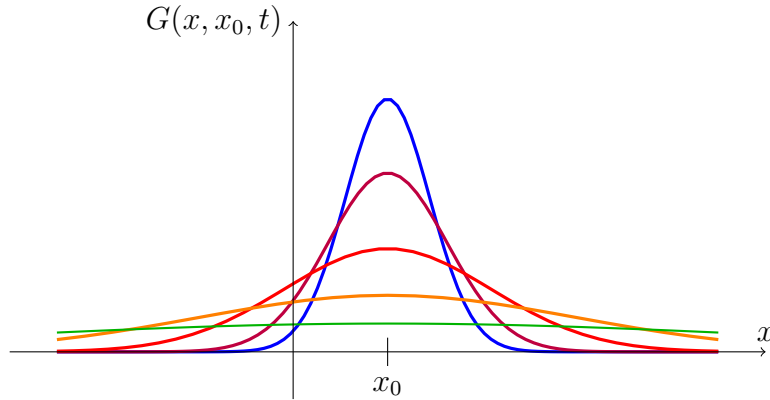
¹²There is once again a mathematician's subtlety in exchanging the derivative and integral when claiming $u_t \mapsto \widehat{u}_t$.

¹³Strictly speaking, it is typically still necessary to specify a “boundary condition at ∞ ” to uniquely characterize solutions. For problems we will consider, requiring $u \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$ is sufficient and will be implicitly assumed.

Example: An important case is $\phi(x) = \delta(x - x_0)$, a concentrated bump of heat at x_0 . Then

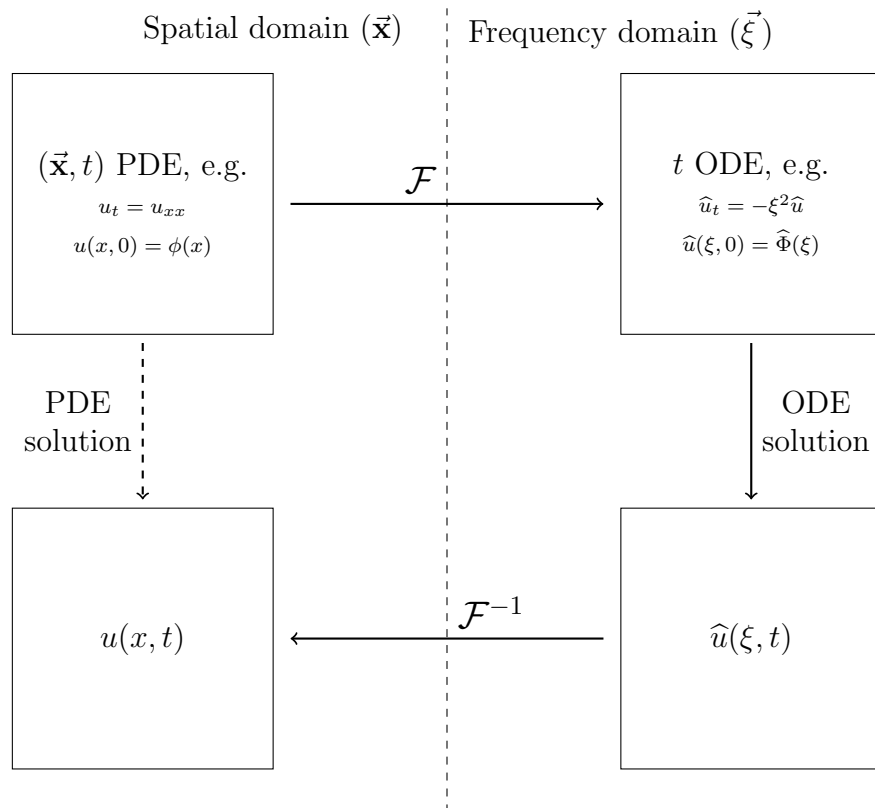
$$\begin{aligned}\widehat{\Phi}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - x_0) e^{-i\xi x} dx = \frac{e^{-i\xi x_0}}{\sqrt{2\pi}} \\ \Rightarrow u(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\xi^2 t} e^{i\xi(x-x_0)} d\xi = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \left[e^{-t\xi^2} \right] (x - x_0) = \boxed{\frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-x_0)^2}{4t}}}\end{aligned}$$

$G(x, x_0, t) := \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-x_0)^2}{4t}}$ is called the system's Green's function, impulse response function, or fundamental solution. The evolution of $G(x, x_0, t)$ with t is shown below; see also [Desmos](#).



□

This exemplifies the procedure by which transform methods proceed, shown schematically below.



In transform problems, one commonly obtains $\widehat{u}$ as a product of known transforms. Indeed, above

$$\widehat{u}(\xi, t) = \widehat{\Phi}(\xi) e^{-\xi^2 t} = \frac{1}{\sqrt{2t}} \mathcal{F}[\phi] \cdot \mathcal{F} \left[e^{-\frac{x^2}{4t}} \right] \quad (34)$$

Unfortunately, it is distinctly *not* the case that $\mathcal{F}^{-1} [\mathcal{F}[f] \cdot \mathcal{F}[g]] = fg$. This presents a question:

$$\mathcal{F}^{-1} [\mathcal{F}[f] \cdot \mathcal{F}[g]] = ?$$

This is answered by introducing the *convolution* operation between two functions $f(x)$, $g(x)$ on $\mathbb{R}$. The convolution $f * g$ of f , g is simply a new function given by

$$(f * g)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

Example: The convolution of $f(x) = x$ and $g(x) = e^{-x^2}$ is

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-y)e^{-y^2}dy = \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2}dy = \frac{x}{\sqrt{2}}$$

□

One can check that this operation is commutative and associative, and it distributes across addition:

$$\begin{aligned} f * g &= g * f \\ (f * g) * h &= f * (g * h) \\ f * (g_1 + g_2) &= f * g_1 + f * g_2 \end{aligned}$$

The following is among the most significant properties of convolution.

Theorem (Convolution Theorem): if f and g are absolutely integrable, then

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g] = \widehat{F}(\xi) \widehat{G}(\xi).$$

Proof.

$$\begin{aligned} \mathcal{F}[f * g](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)(x) e^{-i\xi x} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-y)g(y)dy \right] e^{-i\xi x} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-i\xi x} f(x-y)dx \right] g(y)dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-i\xi(z+y)} f(z)dz \right] g(y)dy \\ &= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi z} f(z)dz \right] \cdot \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi y} g(y)dy \right] = \widehat{F}(\xi) \cdot \widehat{G}(\xi) \end{aligned}$$

□

Convolution is precisely the spatial operation arising from multiplication in the frequency domain.

Hence, we have found

$$\boxed{\mathcal{F}^{-1} [\mathcal{F}[f] \cdot \mathcal{F}[g]] = f * g}$$

Day 12: Fourier Transforms in IVP's Cont'd (Lessons 12, 14 (kind of))

How are general solutions related to the fundamental solution $G(x, x_0, t)$?

Returning to the IVP

$$u_t = u_{xx}, \quad u(x, 0) = \phi(x),$$

recall that we found in (34) that the solution in the frequency domain $\hat{u}(\xi, t)$ is

$$\hat{u}(\xi, t) = \frac{1}{\sqrt{2t}} \mathcal{F}[\phi] \cdot \mathcal{F}\left[e^{-\frac{x^2}{4t}}\right]$$

The Convolution Theorem now tells us that the solution is

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1}[\hat{u}] = \frac{1}{\sqrt{2t}} \mathcal{F}^{-1}\left[\mathcal{F}[\phi] \cdot \mathcal{F}\left[e^{-\frac{x^2}{4t}}\right]\right] = \frac{1}{\sqrt{2t}} (\phi * e^{-\frac{x^2}{4t}}) \\ &= \boxed{\frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(y) e^{-\frac{(x-y)^2}{4t}} dy} \end{aligned}$$

Being non-oscillatory, this integral often has (much) better convergence properties than that in (33).

An illuminating observation: this may be written in terms of the Green's function $G(x, y, t)$ as

$$\boxed{u(x, t) = \int_{-\infty}^{\infty} \phi(y) G(x, y, t) dy}$$

This is just superposition: considering $\phi(x)$ to be the combination of the infinitum of impulses $\phi(y)\delta(x-y)$ at each y , the full response is the sum of the responses to these impulses.

Example: The solution to the IVP

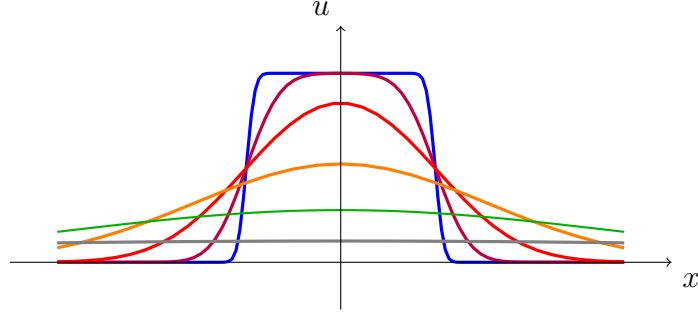
$$u_t = u_{xx}, \quad u(x, 0) = H(x+1) - H(x-1) = \begin{cases} 1 & -1 < x < 1 \\ 0 & |x| > 1 \end{cases}$$

is simply¹⁴

$$\begin{aligned} u(x, t) &= \frac{1}{2\sqrt{\pi t}} \int_{-1}^1 e^{-\frac{(y-x)^2}{4t}} dy = \frac{1}{\sqrt{\pi}} \int_{(-1-x)/2\sqrt{t}}^{(1-x)/2\sqrt{t}} e^{-z^2} dz \\ &= \boxed{\frac{1}{2} \left[\operatorname{erf}\left(\frac{1-x}{2\sqrt{t}}\right) + \operatorname{erf}\left(\frac{1+x}{2\sqrt{t}}\right) \right]} \end{aligned}$$

Snapshots of this evolution are shown below. See also [Desmos](#).

¹⁴Recall the *error function* $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds$.



□

Transform methods readily accommodate inhomogeneous problems. Consider the IVP

$$u_t = u_{xx} + f(x, t), \quad u(x, 0) = \phi(x) \text{ for } x \in \mathbb{R}.$$

Applying $\mathcal{F}$ to both the PDE and the IC yields

$$\hat{u}_t = -\xi^2 \hat{u} + \hat{F}(\xi, t), \quad \hat{u}(\xi, 0) = \hat{\Phi}(\xi) \text{ for } \xi \in \mathbb{R}$$

As usual, we obtain an ODE in t (at each fixed ξ) that can be straightforwardly solved,

$$\hat{u}(\xi, t) = e^{-\xi^2 t} \left[\hat{\Phi}(\xi) + \int_0^t \hat{F}(\xi, s) e^{\xi^2 s} ds \right],$$

and one may recover $u(x, t)$ via the inverse transform $\mathcal{F}^{-1}$. Again leveraging convolution,

$$\boxed{u(x, t) = \int_{-\infty}^{\infty} \phi(y) G(x, y, t) dy + \int_0^t \left[\int_{-\infty}^{\infty} f(y, s) G(x, y, t-s) dy \right] ds} \quad (35)$$

The first integral is the familiar homogeneous solution for the IC. The remainder,

$$v(x, t) := \int_0^t \left[\int_{-\infty}^{\infty} f(y, s) G(x, y, t-s) dy \right] ds,$$

apparently solves

$$v_t = v_{xx} + f(x, t), \quad v(x, 0) = 0 \text{ for } x \in \mathbb{R}.$$

Interestingly, for each $s \in [0, t]$, the quantity in brackets is the solution (evaluated at $t-s$) to

$$w_t = w_{xx}, \quad w(x, 0) = f(x, s) \text{ for } x \in \mathbb{R}.$$

That is, the contribution $v(x, t)$ of the inhomogeneous term $f(x, t)$ to $u(x, t)$ is the accumulation of the homogeneous responses $w(x, t-s)$ to the IC's $f(x, s)$. This is Duhamel's principle.

We revisit the final problem of Day 9, now in an infinite medium. We first exclude convection:

Example: Between $t = 0$ and $t = 1$, dye is injected at a constant rate r at $x = 0$ into a very long, thin tube of initially pure water. What is the concentration $\rho(x, t)$ of the dye?

The appropriate IVP is

$$\rho_t = D\rho_{xx} + rH(1-t)\delta(x), \quad \rho(x, 0) = 0 \text{ for } x \in \mathbb{R}.$$

Applying (35) with $\phi(x) = 0$, $f(x, t) = rH(1-t)\delta(x)$:

$$\begin{aligned} \rho(x, t) &= \int_0^t \left[\int_{-\infty}^{\infty} f(y, s)G(x, y, t-s)dy \right] ds = r \int_0^t H(1-s) \left[\int_{-\infty}^{\infty} \delta(y)G(x, y, t-s)dy \right] ds \\ &= r \int_0^t H(1-s)G(x, 0, t-s)ds = \int_0^{\min(1,t)} G(x, 0, t-s)(rds) \end{aligned}$$

Explicitly, Duhamel's principle says: the dye at (x, t) is found by taking, for each time $s < t$, the small amount of newly injected dye $r \cdot ds$ (if $s < 1$), computing how much diffuses to x from the injection point 0 by time t (so, over the time interval $t-s$), and “summing” over s . Plugging in G ,

$$\begin{aligned} \rho(x, t) &= \frac{r}{\sqrt{4\pi}} \int_0^{\min(1,t)} \frac{e^{-\frac{x^2}{4D(t-s)}}}{\sqrt{D(t-s)}} ds = \frac{r}{\sqrt{4\pi}} \int_{\max(t-1,0)}^t \frac{e^{-\frac{x^2}{4Du}}}{\sqrt{Du}} du \\ &= \boxed{\frac{r}{D\sqrt{4\pi}} \left[2\sqrt{Du}e^{-\frac{x^2}{4Du}} + x\sqrt{\pi} \operatorname{erf}\left(\frac{x}{2\sqrt{Du}}\right) \right] \Big|_{\max(t-1,0)}^t} \end{aligned}$$

One can visualize this result in [this](#) Desmos demo.

□

Now let us consider the same problem with convection.

Example: Between $t = 0$ and $t = 1$, dye is injected at a constant rate r at $x = 0$ into a very long, thin tube of initially pure water flowing with velocity v . What is the concentration $\rho(x, t)$ of the dye?

The appropriate IVP is now

$$\rho_t = D\rho_{xx} - v\rho_x + rH(1-t)\delta(x), \quad \rho(x, 0) = 0 \text{ for } x \in \mathbb{R}.$$

One could proceed in multiple ways. To take the transform approach from scratch, apply $\mathcal{F}$:

$$\hat{\rho}_t = -(D\xi^2 + i\xi v)\hat{\rho} + \hat{F}(\xi, t), \quad \hat{\rho}(\xi, 0) = 0,$$

where

$$\hat{F}(\xi, t) = \mathcal{F}[rH(1-t)\delta(x)] = rH(1-t)\mathcal{F}[\delta(x)] = \frac{r}{\sqrt{2\pi}}H(1-t).$$

Solving the ODE in t , one has

$$\hat{\rho}(\xi, t) = e^{-(D\xi^2 + i\xi v)t} \cdot \frac{r}{\sqrt{2\pi}} \int_0^t H(1-s)e^{(D\xi^2 + i\xi v)s} ds = \frac{r}{\sqrt{2\pi}} \int_0^{\min(1,t)} e^{-(D\xi^2 + i\xi v)(t-s)} ds.$$

Applying $\mathcal{F}^{-1}$, we find

$$\begin{aligned}
\rho(x, t) &= \frac{r}{\sqrt{2\pi}} \int_0^{\min(1, t)} \mathcal{F}^{-1} \left[e^{-D(t-s)\xi^2} \cdot e^{-i\xi v(t-s)} \right] ds \\
&= \frac{r}{\sqrt{2\pi}} \int_0^{\min(1, t)} \mathcal{F}^{-1} \left[e^{-D(t-s)\xi^2} \right] \Big|_{x-v(t-s)} ds \\
&= \frac{r}{\sqrt{4\pi}} \int_0^{\min(1, t)} \frac{e^{-\frac{(x-v(t-s))^2}{4D(t-s)}}}{\sqrt{D(t-s)}} ds = \frac{r}{\sqrt{4\pi}} \int_{\max(t-1, 0)}^t \frac{e^{-\frac{(x-vu)^2}{4Du}}}{\sqrt{Du}} du \\
&= \boxed{\frac{r}{2v} \left[\operatorname{erf} \left(\frac{vu - x}{2\sqrt{Du}} \right) + e^{\frac{vx}{D}} \operatorname{erf} \left(\frac{vu + x}{2\sqrt{Du}} \right) \right] \Big|_{\max(t-1, 0)}^t}
\end{aligned}$$

Alternatively: one could first solve the homogeneous problem with an impulse IC,

$$u_t = Du_{xx} - vu_x, \quad u(x, 0) = \delta(x - x_0) \text{ for } x \in \mathbb{R},$$

to find the Green's function $G(x, x_0, t)$. Applying $\mathcal{F}$,

$$\widehat{u}_t = -(D\xi^2 + i\xi v)\widehat{u}, \quad \widehat{u}(\xi, 0) = \frac{e^{-i\xi x_0}}{\sqrt{2\pi}},$$

so that

$$\begin{aligned}
\widehat{u}(\xi, t) &= \frac{1}{\sqrt{2\pi}} e^{-(D\xi^2 + i\xi v)t - i\xi x_0} = \frac{1}{\sqrt{2\pi}} e^{-Dt\xi^2} \cdot e^{-i\xi(x_0 + vt)} \\
\Rightarrow G(x, x_0, t) = u(x, t) &= \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \left[e^{-Dt\xi^2} \cdot e^{-i\xi(x_0 + vt)} \right] = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \left[e^{-Dt\xi^2} \right] \Big|_{x-x_0-vt} \\
&= \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-x_0-vt)^2}{4Dt}}
\end{aligned}$$

With G , we can simply apply the general result (35) to find ρ , precisely as in the previous example:

$$\begin{aligned}
\rho(x, t) &= r \int_0^t H(1-s) \left[\int_{-\infty}^{\infty} \delta(y) G(x, y, t-s) dy \right] ds = r \int_0^{\min(1, t)} G(x, 0, t-s) ds \\
&= \frac{r}{\sqrt{4\pi}} \int_0^{\min(1, t)} \frac{e^{-\frac{(x-v(t-s))^2}{4D(t-s)}}}{\sqrt{D(t-s)}} ds,
\end{aligned}$$

giving the same integral as found above. One can visualize this result in [this](#) Desmos demo.

You will explore yet another approach in HW5.

□

Day 13: Wave Equation Intro (Lessons 16,19)

How can we model the dynamics of waves?

We now leave behind the heat equation to treat another broad class of physical phenomena: waves.

Waves are essentially characterized by a quantity $u(\vec{x}, t)$ fluctuating about a reference state, e.g.

- Taut string or membrane waves: vertical position fluctuates about an average.
- Sound waves: air pressure fluctuates about atmospheric pressure.
- Waves through solids: lattice particle positions fluctuate about an equilibrium distribution.
- Gravitational waves: the notion of distance between spacetime points fluctuates about a trivial background geometry.

Derivation details vary between applications. General idea: the restoring *force* is proportional to the difference between u at $(\vec{x}, t)$ and nearby points¹⁵. As this difference is quantified by Δu ,

$$\boxed{u_{tt} = c^2 \Delta u} \quad (36)$$

This is the wave equation with speed c , the prototypical hyperbolic PDE.

Optional: Rough derivation for pressure (sound) waves in air. Consider the total mass $M(t)$ of air in a 3D region U . If air flows with velocity vector field $\vec{v}(\vec{x}, t)$ and has density $\rho(\vec{x}, t)$, then¹⁶

$$M'(t) = - \int_{\partial U} (\rho \vec{v}) \cdot d\vec{A} \quad \implies \quad M''(t) \approx - \int_{\partial U} (\rho \vec{a}) \cdot d\vec{A}$$

If the only force driving air flow is its pressure gradient, then¹⁷ $\rho \vec{a} \approx -\vec{\nabla} P$, so

$$M''(t) \approx \int_{\partial U} (\vec{\nabla} P) \cdot d\vec{A} = \int_U (\Delta P) dV.$$

Dividing by the volume V of U and taking the limit $V \rightarrow 0$ gives $\rho_{tt} \approx \Delta P$.

Meanwhile, the ideal gas law says $P = \frac{RT}{M_m} \rho$ (M_m is air's molar mass). If T is constant¹⁸, then

$$P_{tt} = \frac{RT}{M_m} \rho_{tt} = \frac{RT}{M_m} \Delta P,$$

so small pressure fluctuations satisfy the wave equation with speed $c = \sqrt{\frac{RT}{M_m}}$.

Now let $u(x, t)$ be the vertical displacement at x of a taut string with linear density $\rho(x)$ and tension $T(x, t)$. Define $\theta(x, t)$ as the angle made by the string with the horizontal at position x . Notably,

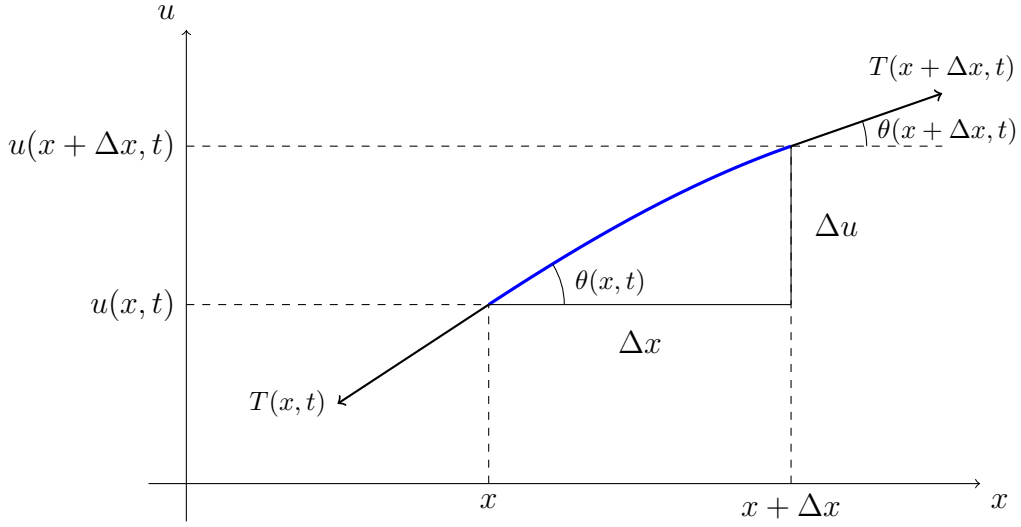
$$\tan(\theta(x, t)) = u_x(x, t).$$

¹⁵For *small* deviations. Most (not all) wave equation derivations invoke some smallness assumption, e.g. to throw away higher order terms in a Taylor expansion.

¹⁶We neglect a $\rho_t \vec{v}$ term in M'' : it is second order in deviations from an equilibrium with $\vec{v}_0 = \vec{0}$ and ρ_0 constant.

¹⁷Another second-order term, $\rho(\vec{v} \cdot \vec{\nabla})\vec{v}$, is dropped here.

¹⁸ T actually does typically vary, minorly adjusting c , but that's an issue for a thermodynamics class.



We apply Newton's second law to a segment of width Δx . The horizontal component reads

$$T(x + \Delta x, t) \cos(\theta(x + \Delta x, t)) - T(x, t) \cos(\theta(x, t)) = 0 \quad \implies \quad (T \cos(\theta))_x = 0$$

Hence, the horizontal tension component $T_h(t) := T \cos(\theta)$ is constant in x .

For the vertical component, note that the segment mass is approximately $\rho(x) \sqrt{(\Delta x)^2 + (\Delta u)^2}$, so

$$\rho(x) \sqrt{(\Delta x)^2 + (\Delta u)^2} u_{tt} \approx T(x + \Delta x, t) \sin(\theta(x + \Delta x, t)) - T(x, t) \sin(\theta(x, t)),$$

and dividing by Δx and taking the limit $\Delta x \rightarrow 0$ gives

$$\rho \sqrt{1 + (u_x)^2} u_{tt} = (T \sin(\theta))_x = (T_h \tan(\theta))_x = T_h (\tan(\theta))_x.$$

Leveraging $u_x = \tan(\theta)$, this reads

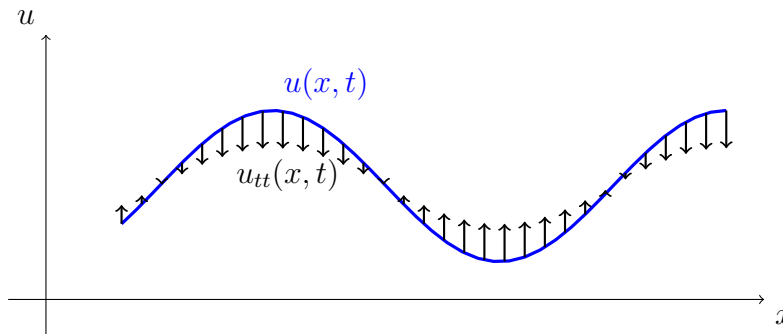
$$\rho \sec(\theta) u_{tt} = T_h u_{xx}$$

Now we must invoke small displacements: for $|\theta| \ll 1$, $\sec(\theta) \approx 1$, so we then have

$$u_{tt} = \frac{T_h}{\rho} u_{xx},$$

and small displacements $u(x, t)$ satisfy the 1D wave equation $u_{tt} = c^2 u_{xx}$ with speed $c = \sqrt{\frac{T_h}{\rho}}$.

This gives rise to a similar picture, emphasizing concavity, to that seen with the 1D heat equation:



Now, however, arrows represent *acceleration* (of the string) rather than *velocity* (of temperature).

As the LHS is acceleration, additional forces (gravity, friction, etc.) simply add to the RHS, e.g.

$$u_{tt} = c^2 u_{xx} - \beta u_t - \gamma u - g$$

has a frictional term βu_t , a restoring force term γu , and a gravitational term g .

It is again natural to pose *initial boundary-value problems*. Common physical boundary conditions:

- Controlled end points:

$$u(0, t) = g_1(t)$$

$$u(L, t) = g_2(t)$$

- Prescribed (vertical) force at ends:

$$u_x(0, t) = g_1(t)$$

$$u_x(L, t) = g_2(t)$$

- Elastic restoring force to a moving equilibrium:

$$T_h u_x(0, t) = k(u(0, t) - g_1(t))$$

$$T_h u_x(L, t) = -k(u(L, t) - g_2(t))$$

While interpretations have changed, we saw these with the heat equation. Broadly, *linear* BC's,

$$\alpha u + \beta \frac{\partial u}{\partial n} = g,$$

are of particular interest. For IC's, we must now specify both $u(x, 0)$ and $u_t(x, 0)$.

Given this data, IBVP's may generally be approached in very much the same way as before.

Day 14: Wave Equation Structure (Lessons 17,18)

What are the qualitative structures of wave solutions?

Before treating IBVP's, let's see how the 1D wave equation yields wave phenomena. Consider

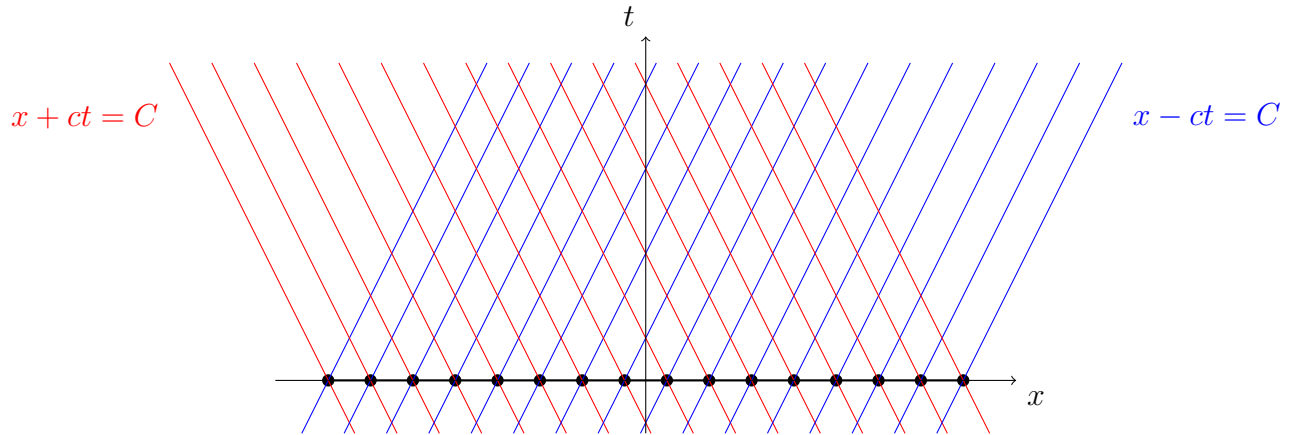
$$u_{tt} = c^2 u_{xx} \quad \Longleftrightarrow \quad (\partial_t^2 - c^2 \partial_x^2)u = 0 \quad \Longleftrightarrow \quad (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0,$$

so $u(x, t)$ will solve the wave equation if *either* $(\partial_t + c\partial_x)u = 0$ or $(\partial_t - c\partial_x)u = 0$.

These are first-order PDE's easily solvable via characteristics: general solutions are

$$\begin{aligned} (\partial_t + c\partial_x)u &= 0 & \Longleftrightarrow & & u(x, t) &= F(x - ct) \\ (\partial_t - c\partial_x)u &= 0 & \Longleftrightarrow & & u(x, t) &= G(x + ct) \end{aligned}$$

Moral: there are two families of characteristics for the 1D wave equation, $x - ct = C$ and $x + ct = C$.



The 1D wave equation has two families of characteristics along which information propagates.

Linearity ensures that combinations of the above solutions also solve the wave equation,

$$\boxed{u(x, t) = F(x - ct) + G(x + ct)}$$

In fact, *all* solutions have this form. To see this, consider the change of coordinates

$$\xi(x, t) := x + ct, \quad \eta(x, t) := x - ct.$$

In the coordinate system (ξ, η) , define the quantity $v(\xi, \eta)$ by

$$v(\xi(x, t), \eta(x, t)) = u(x, t). \tag{37}$$

Taking partial derivatives, one finds

$$\begin{aligned} u_x &= v_\xi \xi_x + v_\eta \eta_x = v_\xi + v_\eta \\ u_t &= v_\xi \xi_t + v_\eta \eta_t = c(v_\xi - v_\eta) \\ u_{xx} &= (u_x)_x = (v_\xi + v_\eta)_\xi \xi_x + (v_\xi + v_\eta)_\eta \eta_x = v_{\xi\xi} + 2v_{\eta\xi} + v_{\eta\eta} \\ u_{tt} &= (u_t)_t = c(v_\xi - v_\eta)_\xi \xi_t + c(v_\xi - v_\eta)_\eta \eta_t = c^2(v_{\xi\xi} - 2v_{\eta\xi} + v_{\eta\eta}) \end{aligned}$$

Hence, restating the wave equation for u in terms of v reads

$$0 = u_{tt} - c^2 u_{xx} = c^2 [(v_{\xi\xi} - 2v_{\eta\xi} + v_{\eta\eta}) - (v_{\xi\xi} + 2v_{\eta\xi} + v_{\eta\eta})] = -4c^2 v_{\eta\xi},$$

giving simply $v_{\eta\xi} = 0$. This may be solved immediately by simply integrating:

$$\begin{aligned} v_{\eta} &= \int v_{\eta\xi} d\xi = \int 0 d\xi = f(\eta) \\ \implies v &= \int v_{\eta} d\eta = \int f(\eta) d\eta = F(\eta) + G(\xi), \end{aligned}$$

where $f(\eta)$ is arbitrary, F is an antiderivative of f , and $G(\xi)$ is arbitrary. Now (37) reads

$$u(x, t) = F(\eta(x, t)) + G(\xi(x, t)) = F(x - ct) + G(x + ct)$$

Hence $u(x, t)$ is always comprised of pieces shifting steadily, either to the right or left, at speed c .

We use this decomposition to solve the wave equation IVP on an infinite domain, $-\infty < x < \infty$:

$$u_{tt} = c^2 u_{xx}, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \quad \text{for } x \in \mathbb{R}$$

We simply impose the IC's on the general form found above:

$$\begin{aligned} \phi(x) &= u(x, 0) = F(x) + G(x), \\ \psi(x) &= u_t(x, 0) = cG'(x) - cF'(x). \end{aligned}$$

Integrating the latter yields, for some constant C ,

$$G(x) - F(x) = \frac{1}{c} \int_0^x \psi(s) ds + C$$

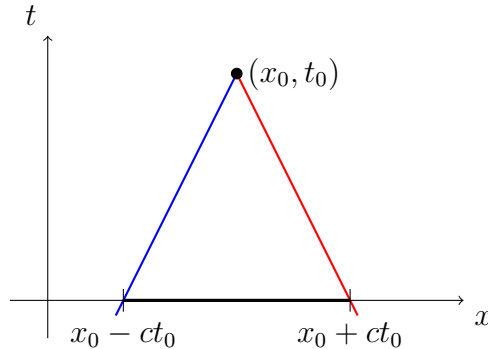
A bit of algebra now yields

$$\begin{aligned} F(x) &= \frac{1}{2} \phi(x) - \frac{1}{2c} \int_0^x \psi(s) ds - \frac{C}{2}, \\ G(x) &= \frac{1}{2} \phi(x) + \frac{1}{2c} \int_0^x \psi(s) ds + \frac{C}{2}, \end{aligned}$$

and plugging into the identity $u(x, t) = F(x - ct) + G(x + ct)$ gives

$$\boxed{u(x, t) = \frac{1}{2} [\phi(x - ct) + \phi(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds} \quad (38)$$

This result is known as the D'Alembert solution. It implies that $u(x_0, t_0)$ depends only on the initial data for $x_0 - ct_0 \leq x \leq x_0 + ct_0$: the wave equation cannot propagate information faster than c .¹⁹



¹⁹This is in stark contrast with the heat equation, under which information propagates infinitely quickly.

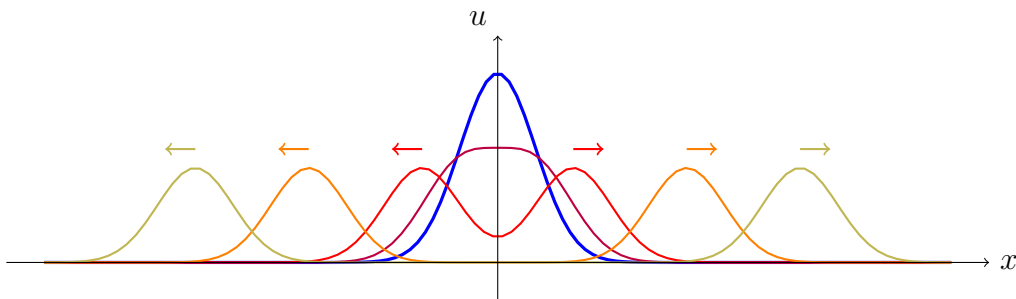
Example: The solution to the IVP

$$u_{tt} = c^2 u_{xx}, \quad u(x, 0) = e^{-x^2}, \quad u_t(x, 0) = 0, \quad \text{for } x \in \mathbb{R}$$

is given immediately by (38),

$$u(x, t) = \frac{1}{2} \left[e^{-(x-ct)^2} + e^{-(x+ct)^2} \right].$$

The initial Gaussian profile is comprised of two pulses moving to the left and right: see [Desmos](#).



□

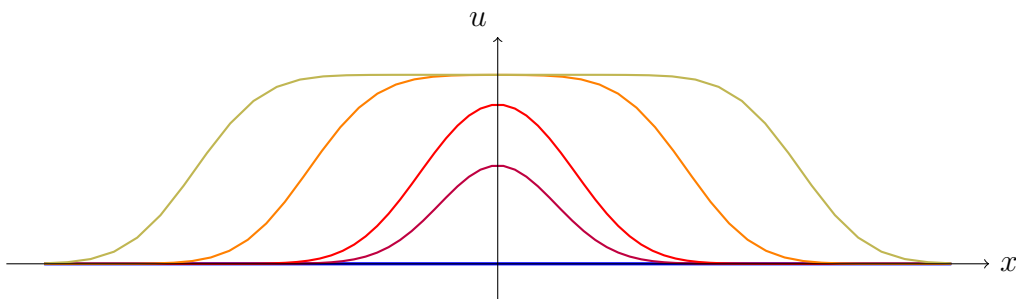
Example: The solution to the IVP

$$u_{tt} = c^2 u_{xx}, \quad u(x, 0) = 0, \quad u_t(x, 0) = e^{-x^2}, \quad \text{for } x \in \mathbb{R}$$

is also given immediately by (38),

$$u(x, t) = \frac{\sqrt{\pi}}{4c} [\operatorname{erf}(x + ct) - \operatorname{erf}(x - ct)].$$

The Gaussian's integral is split into two subtracted pieces moving to the left and right: see [Desmos](#).



Points (x_0, t_0) where u has leveled off (to $\frac{\sqrt{\pi}}{2c}$) have the Gaussian's bump within $(x_0 - ct_0, x_0 + ct_0)$

□

Now consider a *semi*-infinite IBVP, modeling (say) a very long taut string fastened at one end:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0, t) &= 0, \quad 0 \leq t < \infty, \\ u(x, 0) &= \phi(x), \\ u_t(x, 0) &= \psi(x), \quad 0 \leq x < \infty. \end{aligned}$$

It's still true that $u(x, t) = F(x - ct) + G(x + ct)$, and the IC's yield the same algebra as before, so

$$\begin{aligned} F(x) &= \frac{1}{2}\phi(x) - \frac{1}{2c} \int_0^x \psi(s)ds, \\ G(x) &= \frac{1}{2}\phi(x) + \frac{1}{2c} \int_0^x \psi(s)ds, \end{aligned} \quad (39)$$

Since $\phi(x)$, $\psi(x)$ are only defined on $x > 0$, however, these are now only meaningful for $x > 0$.

No issue evaluating $G(x + ct)$: shifting data to the left is unambiguous.

Evaluating $F(x - ct)$ is problematic (for $ct > x$): what is shifting right *into* our domain *from* $x < 0$?

Better question: what *must* be shifting in from $x < 0$ to maintain the BC $u(0, t) = 0$?

Answer: precisely the opposite of what is shifting out *to* $x < 0$. This is achieved by replacing ϕ and ψ in (39) by their *odd extensions* ϕ_o and ψ_o , defined by

$$\phi_o(x) := \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases}$$

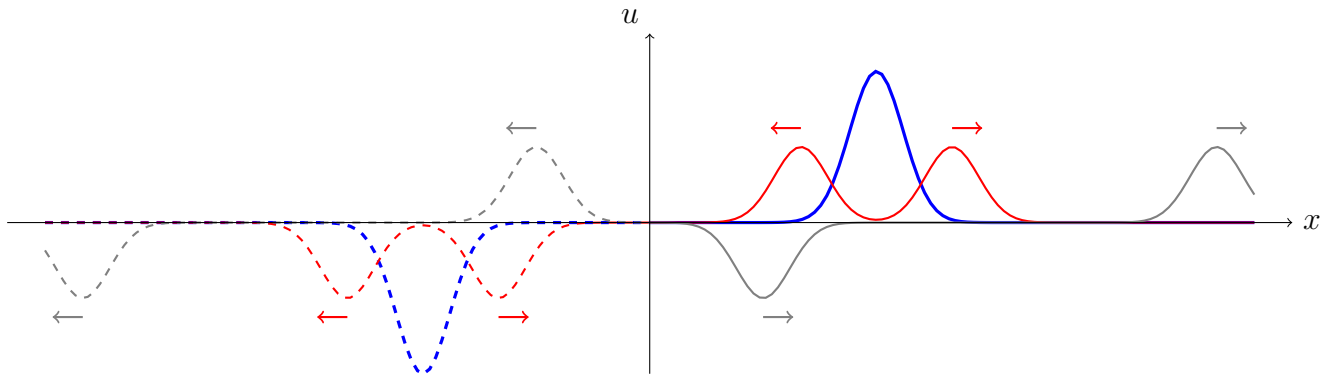
(similarly for ψ_o), to specify F and G everywhere. The semi-infinite IBVP solution is then simply

$$u(x, t) = \frac{1}{2} [\phi_o(x - ct) + \phi_o(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_o(s)ds$$

Example: According to this formula, the solution to the IBVP

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0, t) &= 0, \quad 0 \leq t < \infty, \\ u(x, 0) &= e^{-a(x-2)^2}, \\ u_t(x, 0) &= 0, \quad 0 \leq x \leq \infty, \end{aligned}$$

is given by splitting the odd extension of the Gaussian into pulses moving left and right: see [Desmos](#).



On $x > 0$ (the actual solution), the left-moving pulse “reflects” off of $x = 0$ with a sign change.

□

Day 15: Wave Equation IBVP's (Lesson 20)

How do we solve finite wave IBVP's?

Finally, we consider IBVP's on a finite interval. The prototypical such problem is

$$\begin{aligned}u_{tt} &= c^2 u_{xx}, \\u(0, t) &= 0, \\u(L, t) &= 0, \quad 0 \leq t < \infty, \\u(x, 0) &= \phi(x), \\u_t(x, 0) &= \psi(x), \quad 0 \leq x \leq L.\end{aligned}\tag{40}$$

We may proceed by separation of variables, positing $u(x, t) = X(x)T(t)$, so that the PDE reads

$$X(x)T''(t) = c^2 X''(x)T(t) \implies \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2,$$

giving $T'' + c^2 \lambda^2 T = 0$ and the familiar spatial BVP

$$X'' + \lambda^2 X = 0, \quad X(0) = X(L) = 0,$$

with solutions $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ for $\lambda_n = \frac{n\pi}{L}$. $T_n(t)$ is qualitatively different from before, however:

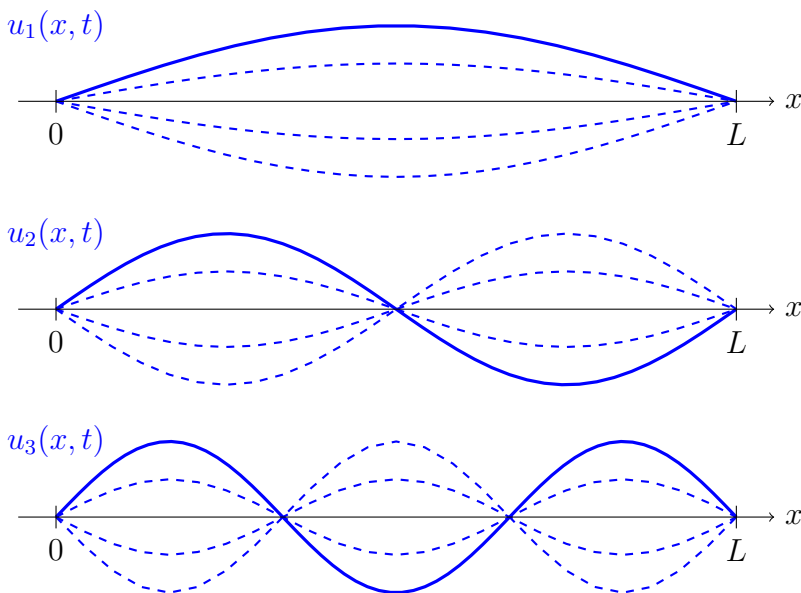
$$T_n(t) = a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right),$$

giving oscillations in time rather than the exponential decay characteristic of heat problems.

The separable solutions to our wave BVP, then, are

$$u_n(x, t) = \left(a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right).$$

Such solutions are often called standing waves, and u_n is sometimes called the n th harmonic.



Note that even though standing waves appear stationary, the identities

$$\begin{aligned}\cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right) &= \frac{1}{2} \left[\sin\left(\frac{n\pi}{L}(x - ct)\right) + \sin\left(\frac{n\pi}{L}(x + ct)\right) \right], \\ \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right) &= \frac{1}{2} \left[\cos\left(\frac{n\pi}{L}(x - ct)\right) - \cos\left(\frac{n\pi}{L}(x + ct)\right) \right]\end{aligned}$$

indicate that these are still comprised of left- and right-moving components.

One can see standing wave solutions dynamically in [this](#) Desmos demo: play the t slider and vary n .

As (40) is homogeneous: superposition! Can combinations of u_n accommodate any initial data?

$$u(x, t) \stackrel{?}{=} \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right) \quad (41)$$

This is straightforward to answer:

$$\begin{aligned}\phi(x) = u(x, 0) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \\ \psi(x) = u_t(x, 0) &= \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right)\end{aligned} \quad \Longleftrightarrow \quad \boxed{\begin{aligned}a_n &= \frac{2}{L} \int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ b_n &= \frac{2}{n\pi c} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx\end{aligned}}$$

As Fourier sine series can represent any “nice” function, (41) can indeed solve (40) for “nice” ϕ, ψ .

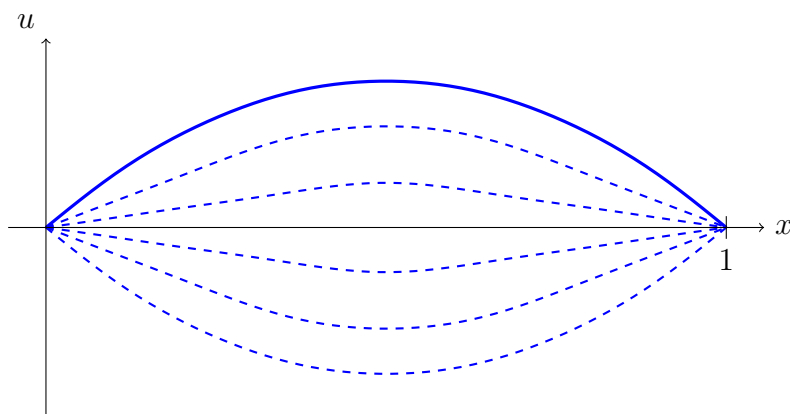
Example: Solve the above IBVP (40) with $L = 1$, $\phi(x) = x(1 - x)$, $\psi(x) = 0$.

The solution takes the form (41) with $b_n = 0$ and

$$a_n = 2 \int_0^1 x(1 - x) \sin(n\pi x) dx = \frac{4}{(n\pi)^3} [1 - (-1)^n]$$

(this was computed on Day 5), so that

$$\boxed{u(x, t) = \frac{8}{\pi^3} \sum_{n=0}^{\infty} \frac{\cos((2n + 1)\pi ct) \sin((2n + 1)\pi x)}{(2n + 1)^3}}$$



One can see this dynamically [here](#). Note the shape distorts over time: this is not a standing wave. $\square$

In this way, the analysis of waves on a finite interval is procedurally similar to the analysis of heat.

Can (40) be solved as $u(x, t) = F(x - ct) + G(x + ct)$? What are F and G outside of $[0, L]$?

HW7: resolved by replacing ϕ and ψ by their *periodic* odd extensions f_o and g_o on $\mathbb{R}$,

$$u(x, t) = \frac{1}{2} [f_o(x - ct) + f_o(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g_o(s) ds$$

Example: See [this](#) Desmos demo of the solution to

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0, t) &= 0, \\ u(1, t) &= 0, & 0 \leq t < \infty, \\ u(x, 0) &= e^{-a(x-1/2)^2}, \\ u_t(x, 0) &= 0, & 0 \leq x \leq 1. \end{aligned}$$

Pulses now effectively reflect off of both boundaries. $\square$

Such a succinct description is atypical when one adjusts BC's and the PDE. Depending on the problem, one can employ SoV or attempt to construct F and G .

Day 16: Canonical Forms (Lesson 23)

How comprehensive are our PDE prototypes?

Recall that the most general second-order linear PDE in two variables (x, y) is

$$\underbrace{Au_{xx} + Bu_{xy} + Cu_{yy}}_{\text{Principal part}} + Du_x + Eu_y + Fu = G, \quad (42)$$

where each of A through G may be functions of (x, y) , and these fall into three basic categories²⁰:

- (1) Parabolic ($B^2 - 4AC = 0$). Prototype: $u_{xx} - u_t = 0$ (heat equation), with $A = 1$, $E = -1$.
- (2) Hyperbolic ($B^2 - 4AC > 0$). Prototype: $u_{xx} - u_{tt} = 0$ (wave equation)²¹, with $A = 1$, $C = -1$.
- (3) Elliptic ($B^2 - 4AC < 0$). Prototype: $u_{xx} + u_{yy} = 0$ (Laplace's equation), with $A = C = 1$.

We will see that for *any* PDE (42), the *principal part* may be made to agree with one of the three prototypes via a coordinate change. When this is achieved, the PDE is said to be in canonical form.

We begin with an arbitrary change of coordinates $\xi(x, y)$, $\eta(x, y)$. Define the quantity $v(\xi, \eta)$ by

$$v(\xi(x, y), \eta(x, y)) = u(x, y).$$

We may apply the multivariable chain rule to relate derivatives of u to those of v :

$$\begin{aligned} u_x &= v_\xi \xi_x + v_\eta \eta_x \\ u_y &= v_\xi \xi_y + v_\eta \eta_y \\ u_{xx} &= v_{\xi\xi} \xi_x^2 + 2v_{\xi\eta} \xi_x \eta_x + v_{\eta\eta} \eta_x^2 + v_\xi \xi_{xx} + v_\eta \eta_{xx} \\ u_{xy} &= v_{\xi\xi} \xi_x \xi_y + v_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + v_{\eta\eta} \eta_x \eta_y + v_\xi \xi_{xy} + v_\eta \eta_{xy} \\ u_{yy} &= v_{\xi\xi} \xi_y^2 + 2v_{\xi\eta} \xi_y \eta_y + v_{\eta\eta} \eta_y^2 + v_\xi \xi_{yy} + v_\eta \eta_{yy} \end{aligned}$$

If u satisfies (42), substituting these identities yields a PDE for $v(\xi, \eta)$ of the form

$$\overline{A}v_{\xi\xi} + \overline{B}v_{\xi\eta} + \overline{C}v_{\eta\eta} + \overline{D}v_\xi + \overline{E}v_\eta + \overline{F}v = \overline{G}.$$

It is a straightforward, if tedious, task to relate the barred quantities $\overline{A}$ - $\overline{G}$ to the original A - G :

$$\begin{aligned} \overline{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ \overline{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ \overline{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\ \overline{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\ \overline{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\ \overline{F} &= F \\ \overline{G} &= G \end{aligned} \quad (43)$$

²⁰If the coefficients are not constant, this categorization may vary throughout the xy plane.

²¹or $u_{\eta\xi} = 0$, with $B = 1$

HW7: verify that the PDE's for u and v have the same type (if $\xi(x, y)$ and $\eta(x, y)$ are real).

Loosely: we have two coordinates to choose, so we may impose two constraints on the principal part.

Recall: the wave equation can be made to have principal part $v_{\xi\eta}$. In general, this would require

$$\begin{aligned} 0 = \bar{A} &\iff 0 = A \left(\frac{\xi_x}{\xi_y} \right)^2 + B \left(\frac{\xi_x}{\xi_y} \right) + C, \\ 0 = \bar{C} &\iff 0 = A \left(\frac{\eta_x}{\eta_y} \right)^2 + B \left(\frac{\eta_x}{\eta_y} \right) + C, \end{aligned} \quad (44)$$

giving two instantiations of the same quadratic equation. Assigning one root to each coordinate,

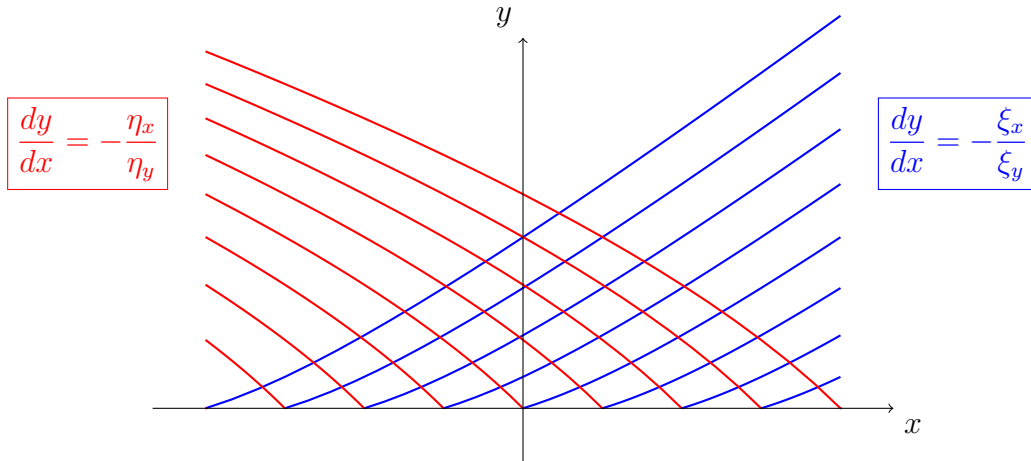
$$\boxed{\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}} \quad (45)$$

These determine the *level sets* of $\xi(x, y)$ and $\eta(x, y)$, curves along which ξ and η are constant.

Indeed, if a curve $\vec{\gamma}(s) := (x(s), y(s))$ satisfies that $\xi(\vec{\gamma}(s))$ is constant, then

$$0 = \frac{d}{ds} [\xi(\vec{\gamma}(s))] = \xi_x \frac{dx}{ds} + \xi_y \frac{dy}{ds} \iff \frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$$

As ξ_x/ξ_y is known, this first-order ODE can, in principle, be solved for such curves in the xy plane.



Characteristics of a hyperbolic PDE, generally curved, are level sets of the new coordinates ξ, η .

This description holds so long as the expressions in (45) are *real*, i.e. if the PDE is *hyperbolic*. These curves are then the PDE's characteristics, and (45) are therefore called the characteristic equations.

Solutions are generally implicit: $f(x, y) = C$. As ξ is constant along these curves, replacing $C \rightarrow \xi$ to obtain $\xi(x, y) = f(x, y)$, and doing similarly for η , yields a choice of ξ, η for which (44) holds.

This choice of ξ, η transforms a hyperbolic PDE into canonical form, having principal part $v_{\xi\eta}$.

If desired, one could obtain the more familiar principal part $w_{tt} - w_{zz}$ by now taking (see Day 14)

$$z(\xi, \eta) = \frac{1}{2}(\xi + \eta), \quad t(\xi, \eta) = \frac{1}{2}(\xi - \eta).$$

Example: Convert the hyperbolic PDE

$$u_{xx} + 5u_{xy} + 4u_{yy} + xu_x = 0$$

to canonical form. What are the canonical coordinates $\xi(x, y)$, $\eta(x, y)$?

Here $A = 1$, $B = 5$, $C = 4$, so $B^2 - 4AC = 5^2 - 4 \cdot 1 \cdot 4 = 9 > 0$, so this is indeed hyperbolic.

The characteristic equations read

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = -1, \quad \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} = -4,$$

and hence characteristics are determined by

$$\begin{aligned} \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = 1 &\implies y = x + C_1 \implies \boxed{\xi(x, y) = y - x} \\ \frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = 4 &\implies y = 4x + C_2 \implies \boxed{\eta(x, y) = y - 4x} \end{aligned}$$

We can now convert from scratch or plug into (43). Of course, $\bar{A} = \bar{C} = 0$ by construction, and

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 2(-1)(-4) + 5(-1 - 4) + 2 \cdot 4 = -9.$$

Similarly $\bar{D} = D\xi_x = -x$, $\bar{E} = D\eta_x = -4x$, $\bar{F} = \bar{G} = 0$. Using $x = (\xi - \eta)/3$, the canonical form is

$$\boxed{v_{\xi\eta} + \frac{\xi - \eta}{27}(v_\xi + 4v_\eta) = 0}$$

We *could* now set $z(\xi, \eta) = (\xi + \eta)/2$ and $t(\xi, \eta) = (\xi - \eta)/2$ and define $w(z, t)$ by

$$w(z(\xi, \eta), t(\xi, \eta)) = v(\xi, \eta)$$

The equivalent PDE for $w(z, t)$ is

$$\boxed{w_{tt} - w_{zz} + \frac{4t}{27}(3w_t - 5w_z) = 0}$$

□

Above, characteristics were straight lines. We now treat a case with curved characteristics.

Example: Convert the PDE²²

$$u_{xx} - xu_{yy} = 0$$

²²rearranging to $u_{yy} = u_{xx}/x$, this is the wave equation with a speed which decreases with x .

to canonical form on the half-plane $x > 0$.

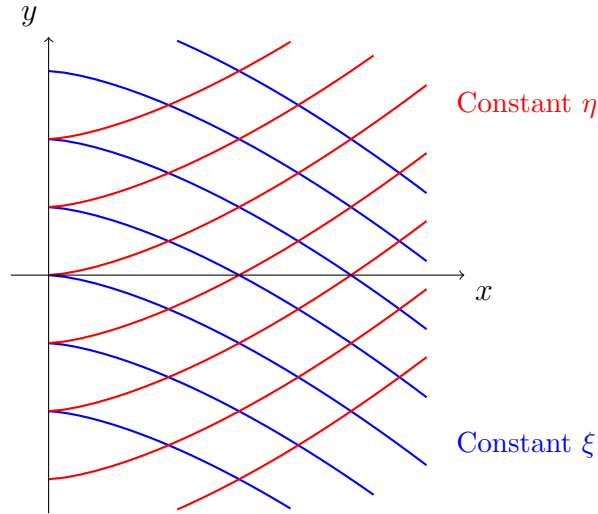
Here $A = 1$, $B = 0$, $C = -x$, so $B^2 - 4AC = 4x > 0$ for $x > 0$, so this is indeed hyperbolic.

The characteristic equations read

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = \sqrt{x}, \quad \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} = -\sqrt{x},$$

and hence characteristics (graphed below) are determined by

$$\begin{aligned} \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = -\sqrt{x} &\implies 3y = -2x^{3/2} + C_1 \implies \boxed{\xi(x, y) = 3y + 2x^{3/2}} \\ \frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \sqrt{x} &\implies 3y = 2x^{3/2} + C_2 \implies \boxed{\eta(x, y) = 3y - 2x^{3/2}} \end{aligned}$$



Note: these map onto the region $\xi > \eta$. Plugging into (43), we find $\bar{A} = \bar{C} = \bar{F} = \bar{G} = 0$ and

$$\begin{aligned} \bar{B} &= 2\xi_x\eta_x - 2x\xi_y\eta_y = -18x - 18x = -36x \\ \bar{D} &= \xi_{xx} = \frac{3}{2\sqrt{x}} \\ \bar{E} &= \eta_{xx} = -\frac{3}{2\sqrt{x}} \end{aligned}$$

Using $x^{3/2} = (\xi - \eta)/4$, the canonical form is then

$$\boxed{v_{\xi\eta} - \frac{v_\xi - v_\eta}{6(\xi - \eta)} = 0}$$

In terms of $z(\xi, \eta) = (\xi + \eta)/2$ and $t(\xi, \eta) = (\xi - \eta)/2$, this becomes²³ (on $t > 0$)

$$\boxed{w_{tt} - w_{zz} + \frac{w_t}{3t} = 0}$$

□

²³This is the wave equation with a *fixed* speed, but with frictional damping that lessens with time.

What about non-hyperbolic equations? Elliptic case: one can follow the procedure, but $\xi(x, y)$ and $\eta(x, y)$ become *complex*, with $\xi^* = \eta$. Characteristic solutions are no longer curves in the xy plane.

For a *real* coordinate transformation, instead take

$$\alpha(x, y) = \frac{1}{2}(\xi(x, y) + \eta(x, y)) = \operatorname{Re}(\xi), \quad \beta(x, y) = \frac{1}{2i}(\xi(x, y) - \eta(x, y)) = \operatorname{Im}(\xi).$$

This yields a PDE with principal part $v_{\alpha\alpha} + v_{\beta\beta}$, as desired.

Example: Convert the PDE

$$u_{xx} + xu_{yy} = 0$$

to canonical form on the half-plane $x > 0$.

Here $A = 1$, $B = 0$, $C = x$, so $B^2 - 4AC = -4x < 0$ for $x > 0$, so this is elliptic.

The characteristic equations read

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = i\sqrt{x}, \quad \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} = -i\sqrt{x},$$

and hence “characteristics” are determined by

$$\begin{aligned} \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = -i\sqrt{x} &\implies 3y = -2ix^{3/2} + C_1 \implies \xi(x, y) = 3y + 2ix^{3/2} \\ \frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = i\sqrt{x} &\implies 3y = 2ix^{3/2} + C_2 \implies \eta(x, y) = 3y - 2ix^{3/2} \end{aligned}$$

As these are complex, we instead utilize the real and imaginary parts,

$$\alpha(x, y) = \operatorname{Re}(\xi) = 3y, \quad \beta(x, y) = \operatorname{Im}(\xi) = 2x^{3/2}.$$

We could plug into (43) (with $\xi, \eta \rightarrow \alpha, \beta$ everywhere). Here, it's easy to proceed from scratch:

$$\begin{aligned} u_x &= v_\alpha \alpha_x + v_\beta \beta_x = 3\sqrt{x}v_\beta \\ u_y &= v_\alpha \alpha_y + v_\beta \beta_y = 3v_\alpha \\ u_{xx} &= \frac{3}{2\sqrt{x}}v_\beta + 9xv_{\beta\beta} \\ u_{yy} &= 9v_{\alpha\alpha} \end{aligned}$$

Plugging into $u_{xx} + xu_{yy} = 0$ directly yields

$$9x(v_{\alpha\alpha} + v_{\beta\beta}) + \frac{3}{2\sqrt{x}}v_\beta = 0.$$

Slightly rearranging and using $\beta = 2x^{3/2}$, the canonical form is

$$\boxed{v_{\alpha\alpha} + v_{\beta\beta} + \frac{v_\beta}{3\beta} = 0}$$

□

Optional: Parabolic case

One now has only one characteristic equation. Defining $\xi(x, y)$ as usual and taking $\eta(x, y)$ to be *anything else*²⁴ (e.g. $\eta(x, y) = y$), one has $\overline{A} = \overline{B} = 0$, and the v PDE reads

$$\overline{C}v_{\eta\eta} + \overline{D}v_{\xi} + \overline{E}v_{\eta} + \overline{F}v = \overline{G}$$

with $\overline{C} \neq 0$. Dividing by $\overline{C}$ gives

Theorem: Any second-order, 2D, linear, parabolic PDE may be cast in the form

$$v_{\eta\eta} + L_1[v] = g(\xi, \eta),$$

for some first-order linear differential operator L_1 .

□

This is the sense in which an arbitrary parabolic PDE is similar to the heat equation.

²⁴“Else” here means η should be independent of ξ , i.e. the transformation $(x, y) \mapsto (\xi, \eta)$ must be (locally) invertible. Of course, this is necessary for it to be meaningfully called a “change of coordinates”.

Day 17: Wave Equation in 2D, 3D (Lesson 24)

How do waves propagate in higher dimensions?

We are well-versed in the solution of hyperbolic and parabolic PDE's in one spatial dimension.

We now treat a higher dimensional problem. Namely, we consider the full-space 3D wave IVP,

$$u_{tt} = c^2 \Delta u = c^2(u_{xx} + u_{yy} + u_{zz}), \quad u(\vec{x}, 0) = \phi(\vec{x}), \quad u_t(\vec{x}, 0) = \psi(\vec{x}), \quad \text{for } \vec{x} \in \mathbb{R}^3 \quad (46)$$

This may be approached via the Fourier transform. Recalling from HW5 that $\mathcal{F}[\Delta u] = -\|\vec{\xi}\|^2 \hat{u}$,

$$\hat{u}_{tt} = -c^2 \|\vec{\xi}\|^2 \hat{u}, \quad \hat{u}(\vec{\xi}, 0) = \hat{\Phi}(\vec{\xi}), \quad \hat{u}_t(\vec{\xi}, 0) = \hat{\Psi}(\vec{\xi}), \quad \text{for } \vec{\xi} \in \mathbb{R}^3.$$

This ODE IVP in t may be straightforwardly solved, yielding

$$\hat{u}(\vec{\xi}, t) = \hat{\Phi}(\vec{\xi}) \cos(c\|\vec{\xi}\|t) + \hat{\Psi}(\vec{\xi}) \frac{\sin(c\|\vec{\xi}\|t)}{c\|\vec{\xi}\|} = \partial_t \left(\hat{\Phi}(\vec{\xi}) \frac{\sin(c\|\vec{\xi}\|t)}{c\|\vec{\xi}\|} \right) + \hat{\Psi}(\vec{\xi}) \frac{\sin(c\|\vec{\xi}\|t)}{c\|\vec{\xi}\|}$$

To find $u(x, t)$, then, we must evaluate

$$\mathcal{F}^{-1} \left[\hat{F}(\vec{\xi}) \frac{\sin(c\|\vec{\xi}\|t)}{c\|\vec{\xi}\|} \right] = ? \quad (47)$$

Toward this end, define the spherical mean of a function f on $\mathbb{R}^3$ to be a new function $M_R[f]$ given at $\vec{x}$ by the mean value of f on the sphere $S_R(\vec{x})$ of radius R centered at $\vec{x}$,

$$M_R[f](\vec{x}) := \frac{1}{4\pi R^2} \int_{\vec{y} \in S_R(\vec{x})} f(\vec{y}) dA.$$

You will show in HW8 that

$$\mathcal{F}[M_R[f]] = \hat{F}(\vec{\xi}) \frac{\sin(\|\vec{\xi}\|R)}{\|\vec{\xi}\|R},$$

so that the inverse transform (47) is simply $tM_{ct}[f]$. We may leverage this to evaluate $\mathcal{F}^{-1}[\hat{u}]$,

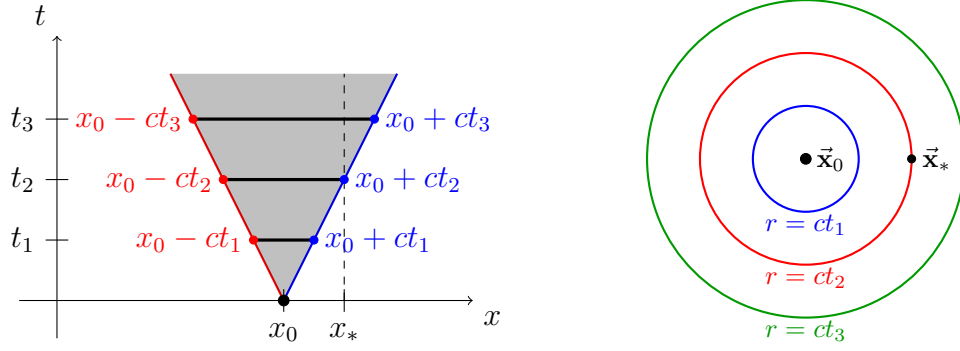
$$\boxed{u(\vec{x}, t) = \partial_t(tM_{ct}[\phi]) + tM_{ct}[\psi]} \quad (48)$$

This is the solution to (46) and 3D analogue of the D'Alembert solution, known as Kirchoff's formula²⁵.

Crucially: u at $(\vec{x}, t)$ is determined by values of ϕ and ψ on the *sphere* centered at $\vec{x}$ with radius ct .

Physically: disturbances travel at *exactly* c in 3D. In 1D, disturbances travelled *no faster* than c .

²⁵This name typically refers the slightly more explicit expression $u(\vec{x}, t) = M_{ct} \left[\phi + ct \frac{\partial \phi}{\partial n} + t\psi \right]$ (with $\frac{\partial}{\partial n}$ the derivative normal to the sphere), but we won't concern ourselves with going this far.



A point-like wave disturbance at $t = 0$ propagates in 1D from x_0 (left) and in 3D²⁶ from $\vec{x}_0$ (right).

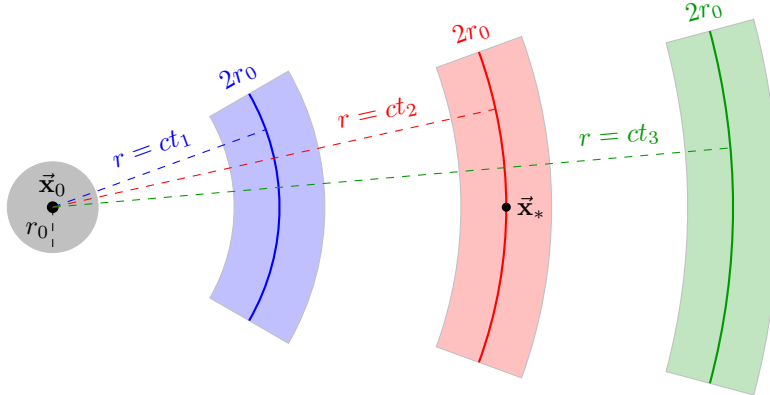
On the left, an observer at x_* which first sees the wave at t_2 continues to see it indefinitely.

On the right, the observer at $\vec{x}_*$ only sees the wave signal *instantaneously* at t_2 .

Waves emanating from a point are shown above. The 3D disturbance is thickened below.

In any dimension, waves have sharp *leading* edges: it is felt suddenly upon reaching the observer.

In 3D (but not 1D), waves also have sharp *trailing* edges: it is suddenly not felt upon passing by. This feature of 3D waves is known as Huygen's principle.



A spherically-concentrated disturbance with radius r_0 propagates from $\vec{x}_0$ in 3D. At each t , the region in space feeling the disturbance is a spherical shell with central radius ct and thickness $\Delta r = 2r_0$.

The observer at $\vec{x}_*$ sees the wave for a short duration as it passes by.

We've solved the full-space wave IVP in 1D and 3D. Now consider the 2D problem,

$$u_{tt} = c^2 \Delta u = c^2 (u_{xx} + u_{yy}), \quad u(\vec{x}, 0) = \phi(\vec{x}), \quad u_t(\vec{x}, 0) = \psi(\vec{x}), \quad \text{for } \vec{x} \in \mathbb{R}^2 \quad (49)$$

Approach via method of descent: a solution to (46) with ϕ, ψ independent of z will also solve (49).

Indeed: 1D, 2D, and 3D waves can all propagate in 3D space. In this context, 1D waves are called plane waves; 2D waves are called cylindrical waves; and 3D waves are called spherical waves.

To solve (49), we simply replace, $\phi, \psi(x, y, z) \rightarrow \phi, \psi(x, y)$ in the solution (48),

$$u(\vec{x}, t) = \partial_t \left(\frac{1}{4\pi c^2 t} \int_{\vec{y} \in S_{ct}(\vec{x})} \phi(\vec{y}) dA \right) + \frac{1}{4\pi c^2 t} \int_{\vec{y} \in S_{ct}(\vec{x})} \psi(\vec{y}) dA$$

²⁶Take note that the circles here represent *spheres* in 3D space.

Writing $\vec{y} = \vec{x} + (\tilde{x}, \tilde{y}, \tilde{z})$ and using $\tilde{z} = \sqrt{(ct)^2 - \tilde{x}^2 - \tilde{y}^2}$ on the sphere of interest, we have

$$dA = \sqrt{1 + \tilde{z}_x^2 + \tilde{z}_y^2} d\tilde{x}d\tilde{y} = \frac{ct}{\sqrt{(ct)^2 - \tilde{x}^2 - \tilde{y}^2}} d\tilde{x}d\tilde{y},$$

with $(\tilde{x}, \tilde{y})$ varying over a 2D disc D_{ct} of radius ct (centered at $\tilde{x} = \tilde{y} = 0$). Writing $\vec{x} = (x, y, z)$,

$$\frac{1}{4\pi c^2 t} \int_{\vec{y} \in S_{ct}(\vec{x})} \psi(\vec{y}) dA = \frac{1}{2\pi c} \int_{D_{ct}} \frac{\psi(x + \tilde{x}, y + \tilde{y})}{\sqrt{(ct)^2 - \tilde{x}^2 - \tilde{y}^2}} d\tilde{x}d\tilde{y} = \frac{1}{2\pi c} \int_0^{2\pi} \int_0^{ct} \frac{\psi(x', y')}{\sqrt{(ct)^2 - \tilde{r}^2}} \tilde{r} d\tilde{r} d\theta,$$

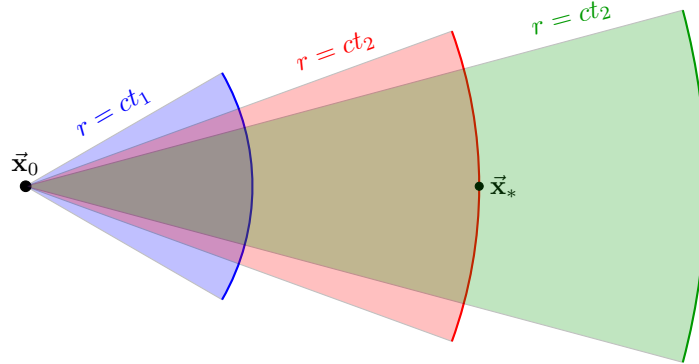
and similarly for ϕ . Here, we've abbreviated $x' := x + \tilde{r} \cos(\theta)$, $y' := y + \tilde{r} \sin(\theta)$.

Hence, the solution to (49) may be written (this is Poisson's formula²⁷)

$$u(x, y, t) = \frac{1}{2\pi c} \left[\int_0^{2\pi} \int_0^{ct} \frac{\psi(x', y')}{\sqrt{(ct)^2 - \tilde{r}^2}} \tilde{r} d\tilde{r} d\theta + \partial_t \left(\int_0^{2\pi} \int_0^{ct} \frac{\phi(x', y')}{\sqrt{(ct)^2 - \tilde{r}^2}} \tilde{r} d\tilde{r} d\theta \right) \right] \quad (50)$$

As an integral over D_{ct} (not its boundary), this 2D solution does *not* respect Huygen's principle.

2D waves decay over time as they pass an observer, but they do not have a sharp trailing edge.



A point-like wave disturbance propagates in 2D from $\vec{x}_0$.

At each t , the region in space feeling the disturbance is a disc with radius ct centered on $\vec{x}_0$.

While it does decay over time, in principle the observer at $\vec{x}_*$ sees the wave indefinitely after t_2 .

This 2D picture generally applies in even dimensions, while the 3D picture applies in odd dimensions (outside of the special case of 1D).

One could further descend to 1D, replacing $\phi, \psi(x, y) \rightarrow \phi, \psi(x)$ to recover the D'Alembert solution.

²⁷Again, this typically refers to a slightly more explicit expression (with numerator evaluated at (x', y')),

$$u(x, y, t) = \frac{1}{2\pi c} \int_0^{2\pi} \int_0^{ct} \frac{\phi + r \frac{\partial \phi}{\partial r} + t\psi}{\sqrt{(ct)^2 - \tilde{r}^2}} \tilde{r} d\tilde{r} d\theta$$

Day 18: Higher dimensional IBVP's (None)

How can we solve IBVP's in higher dimensions?

Consider the prototypical 2D wave IBVP,

$$\begin{aligned} u_{tt} &= c^2 \Delta u, \\ u(0, y, t) &= 0, \quad u(x, 0, t) = 0, \\ u(1, y, t) &= 0, \quad u(x, 1, t) = 0, \quad 0 \leq t < \infty, \\ u(x, y, 0) &= \phi(x, y), \\ u_t(x, y, 0) &= \psi(x, y), \quad 0 \leq x, y \leq 1. \end{aligned} \tag{51}$$

This 2D wave IBVP on a square may model waves on a thin, taut membrane, e.g. a square drumhead.

We may proceed via separation of variables: seek separable solutions $u(x, y, t) = X(x)Y(y)T(t)$,

$$X(x)Y(y)T''(t) = c^2 X''(x)Y(y)T(t) + c^2 X(x)Y''(y)T(t) \iff \frac{T''}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y}$$

The LHS is independent of x, y and the RHS is independent of t , so we must have

$$\frac{T''}{c^2 T} = k, \quad \frac{X''}{X} + \frac{Y''}{Y} = k \iff \frac{Y''}{Y} = k - \frac{X''}{X}.$$

This last equation has a LHS independent of x and RHS independent of y , so we must further have

$$\frac{X''}{X} = l, \quad \frac{Y''}{Y} = k - l.$$

We encounter familiar BVP's for $X(x)$ and $Y(y)$, with no nontrivial solutions if $l > 0$ or $k - l > 0$.

Hence, let us set $l = -\mu^2$, $k - l = -\nu^2$, and $k = -\lambda^2 := -\mu^2 - \nu^2$. Collecting, we've found

$$\begin{aligned} X'' + \mu^2 X &= 0, & X(0) &= X(1) = 0, \\ Y'' + \nu^2 Y &= 0, & Y(0) &= Y(1) = 0, \\ T'' + c^2 \lambda^2 T &= 0, & \lambda^2 &:= \mu^2 + \nu^2. \end{aligned}$$

The BVP's for X and Y are simply two copies of the same problem we've seen repeatedly, yielding

$$X_n(x) = \sin(n\pi x), \quad Y_m(x) = \sin(m\pi x),$$

for $\mu_n = n\pi$ and $\nu_m = m\pi$. Each *combination* of these yields a λ value, $\lambda_{nm}^2 = \pi^2(n^2 + m^2)$, with

$$T_{nm}(t) = a_{nm} \cos(c\lambda_{nm}t) + b_{nm} \sin(c\lambda_{nm}t).$$

We have therefore found that there is a *two*-parameter family of spatially distinct separable solutions,

$$u_{nm}(x, y, t) = [a_{nm} \cos(\lambda_{nm}ct) + b_{nm} \sin(\lambda_{nm}ct)] \sin(n\pi x) \sin(m\pi y).$$

These are 2D standing waves: see [Desmos](#). As usual, we ask if we can build u via superposition,

$$\boxed{u(x, y, t) \stackrel{?}{=} \sum_{n, m=1}^{\infty} [a_{nm} \cos(\lambda_{nm} ct) + b_{nm} \sin(\lambda_{nm} ct)] \sin(n\pi x) \sin(m\pi y)} \quad (52)$$

This reduces to the questions

$$\begin{aligned} \phi(x, y) &= u(x, y, 0) \stackrel{?}{=} \sum_{n, m=1}^{\infty} a_{nm} \sin(n\pi x) \sin(m\pi y), \\ \psi(x, y) &= u_t(x, y, 0) \stackrel{?}{=} \sum_{n, m=1}^{\infty} c\lambda_{nm} b_{nm} \sin(n\pi x) \sin(m\pi y). \end{aligned}$$

This can indeed be achieved for “nice” ϕ, ψ : it is simply a Fourier sine series in each direction.

Namely: as a function of x , we may expand $\phi(x, y)$ in a sine series in x with y -dependent coefficients,

$$\phi(x, y) = \sum_{n=1}^{\infty} a_n(y) \sin(n\pi x), \quad a_n(y) = 2 \int_0^1 \phi(x, y) \sin(n\pi x) dx.$$

As functions of y , each $a_n(y)$ may be expanded in a sine series in y ,

$$\begin{aligned} \phi(x, y) &= \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} a_{nm} \sin(m\pi y) \right] \sin(n\pi x) = \sum_{n, m=1}^{\infty} a_{nm} \sin(n\pi x) \sin(m\pi y), \\ a_{nm} &= 2 \int_0^1 a_n(y) \sin(m\pi y) dy = 4 \int_0^1 \int_0^1 \phi(x, y) \sin(n\pi x) \sin(m\pi y) dx dy. \end{aligned}$$

Applying the same to $\psi(x, y)$, we’ve found that (52) will indeed solve (51) if one takes

$$\boxed{a_{nm} = 4 \int_0^1 \int_0^1 \phi(x, y) \sin(n\pi x) \sin(m\pi y) dx dy, \quad b_{nm} = \frac{4}{c\lambda_{nm}} \int_0^1 \int_0^1 \psi(x, y) \sin(n\pi x) \sin(m\pi y) dx dy}$$

Example: Solve the IBVP (51) with $\phi(x, y) = xy(1-x)(1-y)$ and $\psi(x, y) = 0.1 \sin(2\pi x) \sin(3\pi y)$.

Clearly, $b_{nm} = \frac{\delta_{n,2}\delta_{m,3}}{10c\lambda_{2,3}}$. For a_{nm} , we simply apply the formula to compute

$$\begin{aligned} a_{nm} &= 4 \int_0^1 \int_0^1 \phi(x, y) \sin(n\pi x) \sin(m\pi y) dx dy \\ &= \left[2 \int_0^1 x(1-x) \sin(n\pi x) dx \right] \cdot \left[2 \int_0^1 y(1-y) \sin(m\pi y) dy \right] \\ &= \frac{4}{(n\pi)^3} [1 - (-1)^n] \cdot \frac{4}{(m\pi)^3} [1 - (-1)^m] \\ &= \frac{16}{(nm)^3 \pi^6} [1 - (-1)^n] [1 - (-1)^m]. \end{aligned}$$

Putting these together in (52), we have (watch this evolve in [this](#) Desmos demo)

$$u(x, y, t) = \frac{64}{\pi^6} \sum_{n,m=0}^{\infty} \frac{\sin((2n+1)\pi x) \sin((2m+1)\pi y)}{(2n+1)^3 (2m+1)^3} \cos(\lambda_{2n+1,2m+1} ct) + \frac{\sin(\lambda_{2,3} ct)}{10c\lambda_{2,3}} \sin(2\pi x) \sin(3\pi y)$$

□

In this way, IBVP's on a rectangular domain often reduce to familiar problems, except that there is one spatial BVP in each coordinate direction.

Day 19: Circular Drumhead Problem; Bessel Functions (Lessons 30,31)

How do we solve IBVP's on more general domains?

A realistic drumhead might be modeled on a *circular* domain rather than rectangular, i.e. by

$$\begin{aligned} u_{tt} &= c^2 \Delta u, \\ u(x, y, t) &= 0 \quad \text{for } x^2 + y^2 = 1, \\ u(x, y, 0) &= \phi(x, y), \\ u_t(x, y, 0) &= \psi(x, y), \quad 0 \leq x^2 + y^2 \leq 1. \end{aligned} \tag{53}$$

The separation ansatz $u(x, y, t) = X(x)Y(y)T(t)$ is no longer appropriate, as the BC does not descend to X or Y individually. We can at least separate $u(x, y, t) = U(x, y)T(t)$, which yields

$$\frac{T''}{c^2 T} = \frac{\Delta U}{U} = -\lambda^2$$

(HW8: the separation constant here cannot be positive) and leads to

$$\begin{aligned} \Delta U + \lambda^2 U &= 0, \\ T'' + c^2 \lambda^2 T &= 0. \end{aligned}$$

The T equation is familiar and straightforward, giving the usual temporal oscillations

$$T(t) = A \cos(\lambda c t) + B \sin(\lambda c t).$$

The spatial equation is known as the Helmholtz equation, arising in a variety of settings.

The Helmholtz equation on a domain in $\mathbb{R}^n$ paired with a BC forms an elliptic eigenvalue problem. The breadth of such problems is much richer beyond 1D.

To proceed via our typical methods, we generally must first adapt our coordinate system to the BC.

The IBVP (53) is simplest in polar coordinates (r, θ) on $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, defined by

$$x = r \cos(\theta), \quad y = r \sin(\theta).$$

Recall from HW7: in polar coordinates, the 2D Laplacian $\Delta u = u_{xx} + u_{yy}$ becomes

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta},$$

so the Helmholtz equation and its BC in polar coordinates comprise the BVP

$$U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} + \lambda^2 U = 0, \quad U(1, \theta) = 0$$

This is now amenable to further separation via the ansatz $U(r, \theta) = R(r)\Theta(\theta)$, giving

$$\begin{aligned} 0 &= R''(r)\Theta(\theta) + \frac{1}{r} R'(r)\Theta(\theta) + \frac{1}{r^2} R(r)\Theta''(\theta) + \lambda^2 R(r)\Theta(\theta) \\ \iff -\frac{\Theta''}{\Theta} &= r^2 \frac{R''}{R} + r \frac{R'}{R} + \lambda^2 r^2 = k. \end{aligned}$$

These must be constant (denoted k) in the usual way. We obtain a *periodic* SL BVP in Θ ,

$$\Theta'' + k\Theta = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi).$$

These BC's, necessary for U to make sense, require $k = n^2$ ($n \geq 0$ an integer). Solutions are then

$$\Theta_n(\theta) = A \cos(n\theta) + B \sin(n\theta)$$

Turning to the BVP in R on $0 \leq r \leq 1$, it now reads

$$r^2 R'' + rR' + (\lambda^2 r^2 - n^2)R = 0, \quad R(1) = 0. \quad (54)$$

This *irregular* SL BVP requires an additional BC at $r = 0$: $|R(0)| < \infty$ ²⁸ is sufficient.

This ODE, while famous, is likely unfamiliar. It is a slight variation (if $\lambda \neq 0$) of Bessel's equation,

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0,$$

with $\alpha \in \mathbb{R}$ a parameter²⁹. Solutions generally cannot be expressed in elementary functions.

As the ODE is common, however, a set of linearly independent solutions are named,

$$\begin{aligned} J_\alpha(x) & \text{ order } \alpha \text{ Bessel function of the } \textit{first} \text{ kind,} \\ Y_\alpha(x) & \text{ order } \alpha \text{ Bessel function of the } \textit{second} \text{ kind.} \end{aligned}$$

One may define $J_\alpha(x)$ via the power series resulting from the method of Frobenius,

$$J_\alpha(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \cdot \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha},$$

while $Y_\alpha(x)$ may be defined³⁰ as

$$Y_\alpha(x) := \frac{J_\alpha(x) \cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

See [Desmos](#) for plots. Note that, $J_\alpha(x) \sim x^\alpha$ near $x \rightarrow 0$, meaning $Y_\alpha(x)$ diverges at $x \rightarrow 0$.

Returning to our BVP (54), solutions to the ODE for $R(r)$ are, for each $n \geq 0$,

$$R_n(r) = C_n J_n(\lambda r) + D_n Y_n(\lambda r).$$

The BC $|R(0)| < \infty$ now forces $D_n = 0$, so that our radial solutions are simply $J_n(\lambda r)$.

The BC $R(1) = 0$ stipulates $J_n(\lambda) = 0$. Being oscillatory, J_n has infinitely many roots λ_{nm} .

That is: solutions to the BVP (54) are $R_{nm}(r) = J_n(\lambda_{nm}r)$, with λ_{nm} the m th positive root of J_n .

See [Desmos](#) for plots. The λ_{nm} have no formula, but are well-tabulated and readily computable.

²⁸A distinctly reasonable assumption for our drumhead.

²⁹General theory of Bessel functions allows $\alpha \in \mathbb{C}$ and complex arguments, but we have no need for this generality.

³⁰This appears singular for integer orders $\alpha = n$, but one may simply define $Y_n(x) := \lim_{\alpha \rightarrow n} Y_\alpha(x)$.

	λ_{nm}	n				
		0	1	2	3	4
m	1	2.40	3.83	5.13	6.38	7.59
	2	5.52	7.02	8.42	9.76	11.06
	3	8.65	10.17	11.62	13.02	14.37
	4	11.79	13.32	14.80	16.22	17.62
	5	14.93	16.47	17.96	19.41	20.83
				$\vdots$		

We've found that all polar-separable solutions to the spatial BVP (in both r and θ), then, are

$$U_{nm}(r, \theta) = J_n(\lambda_{nm}r) [A \cos(n\theta) + B \sin(n\theta)].$$

Appending the temporal part, we've obtained all polar-separable solutions to the BVP in (53),

$$u_{nm}(r, \theta, t) = J_n(\lambda_{nm}r) [(A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)) \cos(\lambda_{nm}ct) \\ + (C_{nm} \cos(n\theta) + D_{nm} \sin(n\theta)) \sin(\lambda_{nm}ct)]$$

These are our standing waves, the fundamental modes of a circular drum's vibration³¹: see [Desmos](#).

³¹While this whole mess must be kept when building a general superposition (beware: the text erroneously claims otherwise), the n, m -th mode in isolation has the simpler-looking profile, up to a time shift and rotation, of

$$J_n(\lambda_{nm}r) \cos(n\theta) \cos(\lambda_{nm}ct)$$

Day 20: Circular Drumhead Cont.'d; Fourier-Bessel Series (Lesson 30)

We continue our discussion of the circular drumhead IBVP (53),

$$\begin{aligned} u_{tt} &= c^2 \Delta u, \\ u(x, y, t) &= 0 \quad \text{for } x^2 + y^2 = 1, \\ u(x, y, 0) &= \phi(x, y), \\ u_t(x, y, 0) &= \psi(x, y), \quad 0 \leq x^2 + y^2 \leq 1. \end{aligned}$$

We now ask if we can build $u(r, \theta, t)$ via a superposition of the standing waves found above,

$$\boxed{u(r, \theta, t) \stackrel{?}{=} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} J_n(\lambda_{nm} r) [(A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)) \cos(\lambda_{nm} ct) + (C_{nm} \cos(n\theta) + D_{nm} \sin(n\theta)) \sin(\lambda_{nm} ct)]} \quad (55)$$

This reduces to

$$\begin{aligned} \phi(r, \theta) &= u(r, \theta, 0) \stackrel{?}{=} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} J_n(\lambda_{nm} r) (A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)), \\ \psi(r, \theta) &= u_t(r, \theta, 0) \stackrel{?}{=} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c \lambda_{nm} J_n(\lambda_{nm} r) (C_{nm} \cos(n\theta) + D_{nm} \sin(n\theta)). \end{aligned}$$

This may look somewhat daunting, but there is a familiar core. Let us focus on the first of these,

$$\begin{aligned} \phi(r, \theta) &\stackrel{?}{=} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} J_n(\lambda_{nm} r) (A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)) \\ &= \sum_{n=0}^{\infty} \left[\left(\sum_{m=1}^{\infty} A_{nm} J_n(\lambda_{nm} r) \right) \cos(n\theta) + \left(\sum_{m=1}^{\infty} B_{nm} J_n(\lambda_{nm} r) \right) \sin(n\theta) \right]. \end{aligned}$$

$\phi(r, \theta)$, at each fixed r , has period 2π in θ , and the outer sum over n is apparently its Fourier series. The inner sums over m , then, must yield the familiar Fourier coefficients (23) (with $L = \pi$),

$$\begin{aligned} \sum_{m=1}^{\infty} A_{0m} J_0(\lambda_{0m} r) &=: a_0(r) = \frac{1}{2\pi} \int_0^{2\pi} \phi(r, \theta) d\theta, \\ \sum_{m=1}^{\infty} A_{nm} J_n(\lambda_{nm} r) &=: a_n(r) = \frac{1}{\pi} \int_0^{2\pi} \phi(r, \theta) \cos(n\theta) d\theta, \quad n > 0 \\ \sum_{m=1}^{\infty} B_{nm} J_n(\lambda_{nm} r) &=: b_n(r) = \frac{1}{\pi} \int_0^{2\pi} \phi(r, \theta) \sin(n\theta) d\theta, \quad n > 0 \end{aligned}$$

The functions $a_0(r)$, $a_n(r)$, and $b_n(r)$ are readily computable. How do we proceed to infer A_{nm} , B_{nm} ?

Being the eigenfunctions of the (irregular) SL BVP (54) with distinct eigenvalues, for each fixed n the functions $\{J_n(\lambda_{nm} r)\}_{m=1}^{\infty}$ are orthogonal with respect to the weight $w(r) = r$ on $0 \leq r \leq 1$:

$$\int_0^1 J_n(\lambda_{nm} r) J_n(\lambda_{nk} r) r dr = \delta_{mk} \int_0^1 (J_n(\lambda_{nm} r))^2 r dr = \frac{\delta_{mk}}{2} (J_{n+1}(\lambda_{nm}))^2$$

We must apparently ask: given $f(r)$ on $0 \leq r \leq 1$, can it be expanded in these functions?

Though our SL BVP is irregular, it can be shown that the conclusions of our SL convergence theorem (Day 7) still hold here: one may expand a “nice” (continuously differentiable) $f(r)$,

$$f(r) = \sum_{m=1}^{\infty} C_m J_n(\lambda_{nm} r) \iff C_m = \frac{2}{(J_{n+1}(\lambda_{nm}))^2} \int_0^1 r f(r) J_n(\lambda_{nm} r) dr.$$

This is the Fourier-Bessel series for $f(r)$. Applying this to each of $a_0(r)$, $a_n(r)$, and $b_n(r)$ above,

$$\begin{aligned} A_{0m} &= \frac{2}{(J_1(\lambda_{0m}))^2} \int_0^1 r a_0(r) J_0(\lambda_{0m} r) dr = \frac{1}{\pi (J_1(\lambda_{0m}))^2} \int_0^{2\pi} \int_0^1 \phi(r, \theta) J_0(\lambda_{0m} r) r dr d\theta, \\ A_{nm} &= \frac{2}{(J_{n+1}(\lambda_{nm}))^2} \int_0^1 r a_n(r) J_n(\lambda_{nm} r) dr = \frac{2}{\pi (J_{n+1}(\lambda_{nm}))^2} \int_0^{2\pi} \int_0^1 \phi(r, \theta) J_n(\lambda_{nm} r) \cos(n\theta) r dr d\theta \\ B_{nm} &= \frac{2}{(J_{n+1}(\lambda_{nm}))^2} \int_0^1 r b_n(r) J_n(\lambda_{nm} r) dr = \frac{2}{\pi (J_{n+1}(\lambda_{nm}))^2} \int_0^{2\pi} \int_0^1 \phi(r, \theta) J_n(\lambda_{nm} r) \sin(n\theta) r dr d\theta \end{aligned}$$

These integrals may be taken over the unit disk D_1 , and minor adjustments yield C_{nm} , D_{nm} :

$$\boxed{\begin{aligned} A_{0m} &= \frac{\int_{D_1} \phi(r, \theta) J_0(\lambda_{0m} r) dA}{\pi (J_1(\lambda_{0m}))^2}, & C_{0m} &= \frac{\int_{D_1} \psi(r, \theta) J_0(\lambda_{0m} r) dA}{\pi c \lambda_{0m} (J_1(\lambda_{0m}))^2} \\ A_{nm} &= \frac{2 \int_{D_1} \phi(r, \theta) J_n(\lambda_{nm} r) \cos(n\theta) dA}{\pi (J_{n+1}(\lambda_{nm}))^2}, & C_{nm} &= \frac{2 \int_{D_1} \psi(r, \theta) J_n(\lambda_{nm} r) \cos(n\theta) dA}{\pi c \lambda_{nm} (J_{n+1}(\lambda_{nm}))^2} \\ B_{nm} &= \frac{2 \int_{D_1} \phi(r, \theta) J_n(\lambda_{nm} r) \sin(n\theta) dA}{\pi (J_{n+1}(\lambda_{nm}))^2}, & D_{nm} &= \frac{2 \int_{D_1} \psi(r, \theta) J_n(\lambda_{nm} r) \sin(n\theta) dA}{\pi c \lambda_{nm} (J_{n+1}(\lambda_{nm}))^2} \end{aligned}} \quad (56)$$

With these choices of coefficients, (55) does indeed solve our IBVP (53).

Example: Solve the IBVP (53) with

$$\begin{aligned} \phi(r, \theta) &= J_3(9.76r) \cos(3\theta) - 1.5 J_1(7.02r) \sin(\theta) \\ \psi(r, \theta) &= 3 J_0(2.4r) \end{aligned}$$

We could cite the integrals (56), but here it’s easier to achieve the IC’s with (55) by inspection:

$$u(r, \theta, t) = J_3(9.76r) \cos(3\theta) \cos(9.76ct) - 1.5 J_1(7.02r) \sin(\theta) \cos(7.02ct) + \frac{3}{2.4c} J_0(2.4r) \sin(2.4ct)$$

Warning: this was only so straightforward because 9.76, 7.02, and 2.4 are indeed³² among the λ_{nm} .

One can see this solution evolve in [this](#) Desmos demo.

□

One typically (outside of very special data) leaves the integrals (56) to a computer.

The series (55) is straining for Desmos. [This](#) video shows the evolution of a Gaussian bump.

If interested, you can find the .m MATLAB file which generated this video on Canvas.

³²Of course, these are rounded. The exact values of the roots λ_{nm} are irrational (in fact, transcendental).

Day 21: Elliptic Intro; Harmonic Functions; the Maximum Principle (Lessons 31,32)

How do elliptic problems arise, and what structure do they have?

We now finally turn to elliptic-type PDE's, foremost among these being Laplace's equation,

$$\Delta u = 0. \quad (57)$$

Elliptic problems are very broad, frequently arising in:

- Steady-state phenomena, e.g. electrostatics (recall Day 1), the equilibrium shape of soap film surfaces, or setting $u_t = u_{tt} = 0$ in wave and heat problems.
- Separating variables in dynamical problems, e.g. the wave, heat, and Schrödinger equations.

Such problems have no initial conditions, only boundary conditions: we pose elliptic BVP's.

Namely, elliptic equations are typically posed on a domain $D \subset \mathbb{R}^n$, paired with conditions on ∂D :

- Dirichlet boundary conditions (g is a given function on ∂D),

$$u|_{\partial D} = g.$$

Arise from: fixed temperature (heat) or membrane (waves, soap films); equipotential surfaces (conductors in electrostatics); infinite potential wells (Schrödinger).

- Neumann boundary conditions,

$$\frac{\partial u}{\partial n} \Big|_{\partial D} = g.$$

Arise from: prescribed heat flux (heat), vertical force (waves), surface charge (electrostatics).

- Robin boundary conditions,

$$\left(\alpha u + \beta \frac{\partial u}{\partial n} \right) \Big|_{\partial D} = g.$$

Arise from: environmental heat loss (heat), restoring force to variable equilibrium (waves).

These may be “mixed”, imposing different BC types on different portions of ∂D .

Before solving BVP's, we see some properties of solutions to (57), called harmonic functions.

Consider the spherical mean $M_R[u](\vec{x}_0)$ of a 3D³³ function u at a fixed point $\vec{x}_0$,

$$M_R[u](\vec{x}_0) = \frac{1}{4\pi R^2} \int_{\vec{y} \in S_R(\vec{x}_0)} u(\vec{y}) dA = \frac{1}{4\pi} \int_{\hat{n} \in S_1(\vec{0})} u(\vec{x}_0 + R\hat{n}) dA.$$

We have changed variables via $\vec{y} =: \vec{x}_0 + R\hat{n}$ to integrate over the *unit vectors* $\hat{n}$. Differentiating,

$$\begin{aligned} \frac{d}{dR} (M_R[u](\vec{x}_0)) &= \frac{1}{4\pi} \int_{\hat{n} \in S_1(\vec{0})} \frac{d}{dR} [u(\vec{x}_0 + R\hat{n})] dA = \frac{1}{4\pi} \int_{\hat{n} \in S_1(\vec{0})} \hat{n} \cdot \vec{\nabla} u(\vec{x}_0 + R\hat{n}) dA \\ &= \frac{1}{4\pi R^2} \int_{\vec{y} \in S_R(\vec{x}_0)} \left(\vec{\nabla} u(\vec{y}) \right) \cdot \vec{dA} = \frac{1}{4\pi R^2} \int_{B_R(\vec{x}_0)} (\Delta u) dV. \end{aligned}$$

³³The argument and conclusion here carries over nearly exactly to any dimension.

Since $M_R[u](\vec{x}_0) \rightarrow u(\vec{x}_0)$ as $R \rightarrow 0$, this formalizes our intuition that Δu quantifies the difference between $u(\vec{x}_0)$ and its mean value $M_R[u](\vec{x}_0)$. In the special case that u is harmonic, we've found:

$$\frac{d}{dR} (M_R[u](\vec{x}_0)) = 0 \quad \implies \quad M_R[u](\vec{x}_0) \text{ is independent of } R.$$

Hence, the mean value of u on every sphere centered at $\vec{x}_0$ is the same, $M_R[u](\vec{x}_0) = M_{R'}[u](\vec{x}_0)$!

Taking the limit $R' \rightarrow 0$ yields a structure of harmonic functions called the mean value property,

$$u(\vec{x}_0) = M_R[u](\vec{x}_0).$$

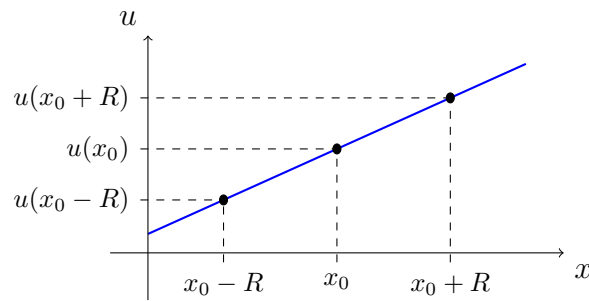
That is: the mean value of u on any sphere is equal to its value at the center.

Example: The simplest manifestation of the MVP is in 1D harmonic functions, which are *linear*:

$$u(x) = C_1 x + C_2$$

Here, the MVP simply says that averaging points shifted right and left from x_0 by R returns $u(x_0)$,

$$u(x_0) = \frac{1}{2} [u(x_0 - R) + u(x_0 + R)].$$



□

The MVP is less trivial beyond 1D: harmonic functions are much broader than linear functions.

Example: The function $u(x, y) = x^2 - y^2$ is harmonic: see [Desmos](#) for an illustration of the MVP.

□

Example: In polar coordinates, $x^2 - y^2 = r^2(\cos^2(\theta) - \sin^2(\theta)) = r^2 \cos(2\theta)$.

HW8: $u(r, \theta) = r^n [A \cos(n\theta) + B \sin(n\theta)]$ is harmonic for any integer n (positive, negative³⁴, or 0).

The MVP for these examples is illustrated in [this](#) Desmos demo.

□

A related, but broader, feature of harmonic functions is known as the (strong) maximum principle:

Theorem: If u is harmonic on a connected, bounded domain $D \subset \mathbb{R}^n$ and attains its maximum value in the interior of D , then u must be *constant* on D .

³⁴For the MVP to hold, u must be harmonic everywhere in $B_R(\vec{x}_0)$, the *ball* of radius R centered at $\vec{x}_0$. For $n < 0$, $u(r, \theta) = r^n [A \cos(n\theta) + B \sin(n\theta)]$ is singular at $r = 0$, so the MVP need not hold if the ball includes the origin.

Proof. This follows rather directly from the mean value property. If u attains its maximum at $\vec{x}_0 \in D$, meaning $u(\vec{x}) \leq u(\vec{x}_0)$ for each $\vec{x} \in D$, then since $u(\vec{x}_0)$ is equal to its mean value on any sphere centered on $\vec{x}_0$ (and contained in D), this means $u(\vec{x}) = u(\vec{x}_0)$ on any such sphere. This forces u to be constant at points near $\vec{x}_0$. One can apply this reasoning again to these new points (as they also have u at its maximum) to “spread” the assurance of constancy throughout D ³⁵. $\square$

An implication: if u is *not* constant, it must achieve its maximum *on the boundary* ∂D .

You are encouraged to review the MVP Desmos demos again with this implication in mind.

These results are also true of minimum values. This yields an essential feature of common BVP’s:

Theorem: There exists *at most one* solution to the inhomogeneous Dirichlet problem

$$\Delta u = f, \quad u|_{\partial D} = g.$$

Proof. If u_1, u_2 are two solutions, then $\bar{u} := u_1 - u_2$ satisfies the homogeneous problem

$$\Delta \bar{u} = 0, \quad \bar{u}|_{\partial D} = 0.$$

That is, $\bar{u}$ is harmonic and vanishes on the boundary. Since both the maximum and minimum values of $\bar{u}$ are achieved on the boundary, $\bar{u} = 0$ everywhere in D , and hence $u_1 = u_2$. $\square$

HW9: This is also true of the corresponding Robin problem. In the Neumann problem, any two solutions must differ by a constant.

The MP can be strengthened: non-constant harmonic functions never have local extrema.

Intuition:³⁶ if $u(x, y)$ had a local maximum at (x_0, y_0) , we generally expect³⁷ $u_{xx}, u_{yy} < 0$ at (x_0, y_0) , but this is not possible if $0 = \Delta u = u_{xx} + u_{yy}$.

³⁵This is a bit hand-wavy: rigorous proof requires some notions from topology, which are beyond this course.

³⁶This is stated for 2D for brevity, but clearly carries over to any dimension

³⁷This is not airtight: maxima can have $u_{xx} = u_{yy} = 0$. A bit of complex analysis is needed to fill the gap.

Day 22: Laplace's Equation in 2D: Circular geometry (Lessons 33,34)

How can we solve elliptic BVP's?

We first consider the Dirichlet problem of Laplace's equation on a 2D disk with radius 1,

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \Delta u = 0 \quad \text{for } 0 \leq r \leq 1, \quad u(1, \theta) = g(\theta) \quad (58)$$

This has a variety of interpretations. It may describe:

- The equilibrium shape of a soap film spread across a wire that's bent from a perfectly circular shape by vertical distance $g(\theta)$ at each position θ .
- The electric potential inside a circle on which the potential $g(\theta)$ is known.
- The equilibrium temperature inside a disk with prescribed temperature $g(\theta)$ on the boundary.

As the PDE is the Helmholtz equation with $\lambda = 0$, we saw on Day 19 that $u(r, \theta) = R(r)\Theta(\theta)$ yields

$$\begin{aligned} \Theta_n(\theta) &= A \cos(n\theta) + B \sin(n\theta), \\ 0 &= r^2 R'' + r R' - n^2 R. \end{aligned}$$

HW8: independent radial solutions are $R(r) = r^n, r^{-n}$ for $n \neq 0$ and $R(r) = 1, \ln(r)$ for $n = 0$.

In applications, regularity at $r = 0$ rules out r^{-n} and $\ln(r)$, so separable solutions of interest are

$$u_n(r, \theta) = r^n [a_n \cos(n\theta) + b_n \sin(n\theta)] \quad (\text{see [Desmos](#)})^{38}$$

As the PDE is homogeneous, we ask whether the solution to (58) can be obtained via superposition,

$$u(r, \theta) \stackrel{?}{=} \sum_{n=0}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)] \quad (59)$$

That is, we ask whether such a superposition can accommodate the boundary condition $g(\theta)$:

$$g(\theta) = u(1, \theta) \stackrel{?}{=} \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

Of course, it can: this is the Fourier series of $g(\theta)$. Recalling (23) (with $L = \pi$), the coefficients are

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta, \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos(n\theta) d\theta, \quad n > 0, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin(n\theta) d\theta, \quad n > 0. \end{aligned} \quad (60)$$

³⁸Compare with images of actual soap films [here](#) and [here](#).

Example: The shape of a soap film spread across a wire bent according to

$$g(\theta) = \sin(\theta) + 0.3 \cos(2\theta) - \cos(3\theta)$$

is simply (see [Desmos](#))

$$u(r, \theta) = r \sin(\theta) + 0.3r^2 \cos(2\theta) - r^3 \cos(3\theta)$$

□

We use (60) to express (59) in a simpler form. First: if the disk had radius R , (59) would read

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

with the same formulas (60). Substituting these in,

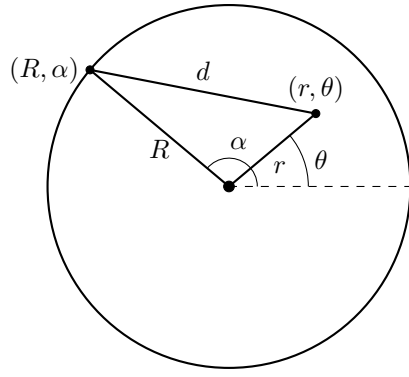
$$\begin{aligned} u(r, \theta) &= \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)] \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) d\alpha + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \int_0^{2\pi} g(\alpha) [\cos(n\alpha) \cos(n\theta) + \sin(n\alpha) \sin(n\theta)] d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos(n(\alpha - \theta)) \right] d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) \left[1 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (e^{in(\alpha - \theta)} + e^{-in(\alpha - \theta)}) \right] d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) \left[1 + \sum_{n=1}^{\infty} \left(\left(\frac{r}{R} e^{i(\alpha - \theta)}\right)^n + \left(\frac{r}{R} e^{-i(\alpha - \theta)}\right)^n \right) \right] d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) \left[1 + \frac{r e^{i(\alpha - \theta)}}{R - r e^{i(\alpha - \theta)}} + \frac{r e^{-i(\alpha - \theta)}}{R - r e^{-i(\alpha - \theta)}} \right] d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) \left[\frac{R^2 - r^2}{R^2 - 2rR \cos(\alpha - \theta) + r^2} \right] d\alpha \end{aligned}$$

This final result is known as the Poisson integral formula,

$$\boxed{u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) \left[\frac{R^2 - r^2}{R^2 - 2rR \cos(\alpha - \theta) + r^2} \right] d\alpha} \quad (61)$$

While typically not analytically evaluable, this can often be more efficiently computed than (59).

$u(r, \theta)$ is apparently a *weighted average* of the boundary values $g(\alpha)$, with weight the quantity in brackets (the Poisson kernel). The denominator is the squared distance from (r, θ) to (R, α) :



Law of cosines:

$$d^2 = R^2 + r^2 - 2rR \cos(\alpha - \theta)$$

When averaging to compute $u(r, \theta)$, then, (61) heavily weights nearby points on the boundary.

To visualize the Poisson kernel's weighting, see [this](#) Desmos demo (the sliders are $s := \frac{r}{R}$, $a := \theta$).

Example: Compare the outputs of (59) and (61) in [Desmos](#) for $g(\theta)$ given on $0 \leq \theta \leq 2\pi$ by

$$g(\theta) = \text{sgn}(\theta - \pi) = \begin{cases} -1 & 0 < \theta < \pi \\ 1 & \pi < \theta < 2\pi \end{cases}$$

In fact, (61) can (with some difficulty) be evaluated exactly in this particular case, giving

$$u(r, \theta) = \text{sgn}(\theta - \pi) + \frac{2}{\pi} \tan^{-1} \left(\frac{1 - r^2}{2r \sin(\theta)} \right)$$

□

We may also pose elliptic problems on *unbounded* domains (common in electrostatics). In particular, we could consider the same problem on the *exterior* of a disk with radius R ,

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \Delta u = 0 \quad \text{for } r \geq R, \quad u(R, \theta) = g(\theta) \quad (62)$$

Separable solutions are the same, but we now typically rule out those diverging at $r \rightarrow \infty$, leaving

$$u_n(r, \theta) = \left(\frac{r}{R} \right)^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

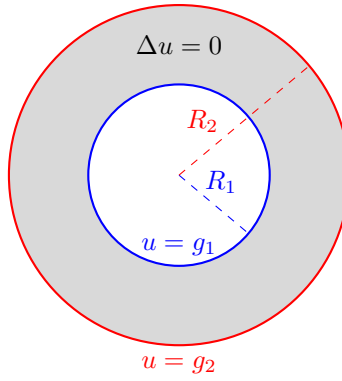
That is, the solution to (62) is simply (with a_0 , a_n , b_n all the same as before, given by (60))

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R} \right)^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

Reviewing the derivation, Poisson's formula (61) still clearly holds, but with r , R switching roles.

Finally, perhaps the broadest problem in this vein is posed on an *annulus*, $R_1 \leq r \leq R_2$,

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \Delta u = 0 \quad \text{for } R_1 \leq r \leq R_2, \quad \begin{aligned} u(R_1, \theta) &= g_1(\theta) \\ u(R_2, \theta) &= g_2(\theta) \end{aligned} \quad (63)$$



We now have no reason to throw out any separable solutions. Indeed, all will be necessary:

$$u(r, \theta) = a_0 + b_0 \ln(r) + \sum_{n=1}^{\infty} [(a_n r^n + b_n r^{-n}) \cos(n\theta) + (c_n r^n + d_n r^{-n}) \sin(n\theta)] \quad (64)$$

We must now choose the coefficients such that both BC's are met:

$$\begin{aligned} g_1(\theta) = u(R_1, \theta) &\stackrel{?}{=} a_0 + b_0 \ln(R_1) + \sum_{n=1}^{\infty} [(a_n R_1^n + b_n R_1^{-n}) \cos(n\theta) + (c_n R_1^n + d_n R_1^{-n}) \sin(n\theta)] \\ g_2(\theta) = u(R_2, \theta) &\stackrel{?}{=} a_0 + b_0 \ln(R_2) + \sum_{n=1}^{\infty} [(a_n R_2^n + b_n R_2^{-n}) \cos(n\theta) + (c_n R_2^n + d_n R_2^{-n}) \sin(n\theta)] \end{aligned}$$

While the coefficients are not directly our a_n, b_n, c_n, d_n , these are simply Fourier series, giving

$$\begin{aligned} a_0 + b_0 \ln(R_1) &= \frac{1}{2\pi} \int_0^{2\pi} g_1(\theta) d\theta, & a_0 + b_0 \ln(R_2) &= \frac{1}{2\pi} \int_0^{2\pi} g_2(\theta) d\theta, \\ a_n R_1^n + b_n R_1^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_1(\theta) \cos(n\theta) d\theta, & a_n R_2^n + b_n R_2^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_2(\theta) \cos(n\theta) d\theta, \\ c_n R_1^n + d_n R_1^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_1(\theta) \sin(n\theta) d\theta, & c_n R_2^n + d_n R_2^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_2(\theta) \sin(n\theta) d\theta. \end{aligned} \quad (65)$$

These constraints may be solved algebraically for the $a_0, b_0, a_n, b_n, c_n, d_n$ with which (64) solves (63).

Example: Consider the annulus problem (63) with $g_1(\theta) = g_2(\theta) = \sin(\theta)$ at $R_1 = 1$ and $R_2 = 2$.

All of the integrals (65) vanish except the last row for $n = 1$, giving $a_0 = b_0 = a_n = b_n = 0$ and

$$c_1 + d_1 = 1, \quad 2c_1 + 2^{-1}d_1 = 1 \quad \implies \quad c_1 = 1/3, \quad d_1 = 2/3,$$

so that (64) reads (see [Desmos](#))

$$u(r, \theta) = \frac{1}{3} \left(r + \frac{2}{r} \right) \sin(\theta)$$

□

Example: Now take $g_1(\theta) = 1$, $g_2(\theta) = 2$ at $R_1 = 1$ and $R_2 = 4$.

All of the integrals (65) vanish except the first row, giving

$$a_0 = 1, \quad a_0 + b_0 \ln(4) = 2 \quad \implies \quad a_0 = 1, \quad b_0 = 1/\ln(4),$$

so that (64) reads (see [Desmos](#))

$$u(r, \theta) = 1 + \frac{\ln(r)}{\ln(4)}$$

□

Day 23: Laplace's Equation in 3D: Spherical geometry (Lesson 35)

How do we solve problems tailored to spherical geometry?

In 3D settings, spherical geometry is foundational. In spherical coordinates (r, θ, φ) defined by

$$z = r \cos(\theta), \quad x = r \sin(\theta) \cos(\varphi), \quad y = r \sin(\theta) \sin(\varphi),$$

one can show that the Laplacian $\Delta u = u_{xx} + u_{yy} + u_{zz}$ may be written

$$\begin{aligned} \Delta u &= u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}(u_{\theta\theta} + \cot(\theta)u_\theta) + \frac{1}{r^2 \sin^2(\theta)}u_{\varphi\varphi} \\ &= \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2 \sin(\theta)}(\sin(\theta)u_\theta)_\theta + \frac{1}{r^2 \sin^2(\theta)}u_{\varphi\varphi}, \end{aligned}$$

so that we may approach the Dirichlet problem inside the unit sphere $0 \leq r \leq 1$,

$$\Delta u = 0 \quad \text{for } 0 \leq r \leq 1, \quad u(1, \theta, \varphi) = g(\theta, \varphi), \quad (66)$$

via separation of variables. First separating the radial dependence, $u(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$, yields

$$\frac{1}{R}(r^2 R')' + \frac{1}{\sin(\theta)Y}(\sin(\theta)Y_\theta)_\theta + \frac{1}{\sin^2(\theta)Y}Y_{\varphi\varphi} = 0.$$

In the usual way, we have

$$\frac{1}{R}(r^2 R')' = -\frac{1}{\sin(\theta)Y}(\sin(\theta)Y_\theta)_\theta - \frac{1}{\sin^2(\theta)Y}Y_{\varphi\varphi} = k$$

for some constant k . Further separating the angular part, $Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi)$ yields

$$\frac{\sin(\theta)}{\Theta}(\sin(\theta)\Theta')' + k \sin^2(\theta) = -\frac{\Phi''}{\Phi} = m^2,$$

for some other constant m^2 : as seen in 2D, this must be a square integer by 2π -periodicity in φ ,

$$\Phi_m(\varphi) = A \cos(m\varphi) + B \sin(m\varphi).$$

The Θ equation, for which we seek nontrivial solutions on $0 \leq \theta \leq \pi$, now reads

$$\Theta'' + \cot(\theta)\Theta' + \left(k - \frac{m^2}{\sin^2(\theta)}\right)\Theta = 0. \quad (67)$$

That is, separation of variables has yielded the following radial, azimuthal, and polar ODE's

$$\begin{aligned} 0 &= r^2 R'' + 2r R' - kR, \\ 0 &= \Phi'' + m^2 \Phi, \\ 0 &= \Theta'' + \cot(\theta)\Theta' + \left(k - \frac{m^2}{\sin^2(\theta)}\right)\Theta. \end{aligned}$$

We have already solved the Φ equation, constraining m to be an integer. We turn to the Θ equation.

This ODE is best understood by setting $x := \cos(\theta)$ and defining $y(x)$ by $y(x(\theta)) := \Theta(\theta)$, so

$$\Theta' = y' \cdot x_\theta = -\sin(\theta)y', \quad \Theta'' = \sin^2(\theta)y'' - \cos(\theta)y',$$

yielding an ODE for y , posed on $-1 \leq x \leq 1$, reading

$$(1 - x^2)y'' - 2xy' + \left(k - \frac{m^2}{1 - x^2}\right)y = 0.$$

We seek nontrivial solutions to this ODE, the general Legendre equation, satisfying $|y(\pm 1)| < \infty$.

This is an *irregular* SL BVP. We first consider the case $m = 0$ (corresponding to no φ dependence):

$$(1 - x^2)y'' - 2xy' + ky = 0, \quad |y(\pm 1)| < \infty$$

This is Legendre's equation. As with Bessel's equation, one may construct power series solutions.

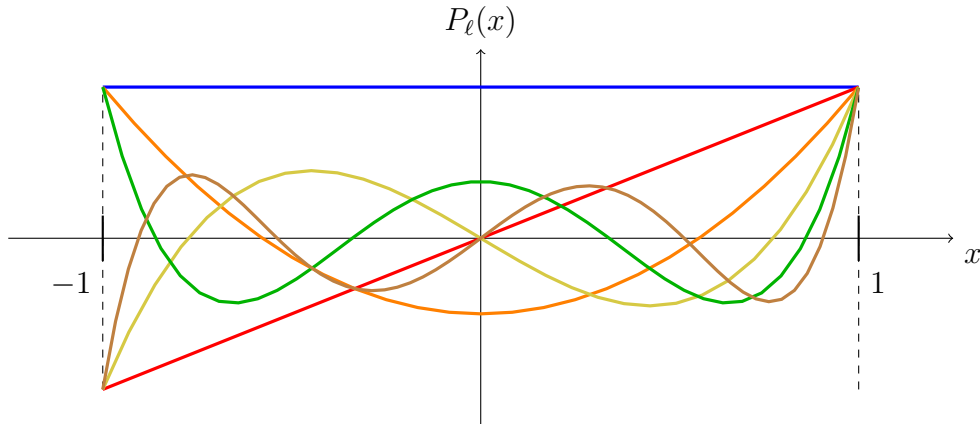
The essential result: only for $k = \ell(\ell + 1)$, $\ell \in \mathbb{Z}$, do there exist nontrivial solutions regular at both $x = \pm 1$, and in this case the eigenfunction's power series *terminates*, leaving a *polynomial*.

These eigenfunctions are known as the Legendre polynomials $P_\ell(x)$, given by Rodrigues' formula

$$P_\ell(x) = \frac{1}{2^\ell \cdot \ell!} \frac{d^\ell}{dx^\ell} [(x^2 - 1)^\ell], \quad \ell = 0, 1, 2, \dots$$

P_ℓ is a degree- ℓ polynomial with ℓ roots on $-1 \leq x \leq 1$, scaled to $P_\ell(1) = 1$. The first several are:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\ &\vdots \end{aligned}$$



Hence, $m = 0$ solutions to (67) are $\Theta_\ell(\theta) = P_\ell(\cos(\theta))$. The radial ODE with $k \rightarrow \ell(\ell + 1)$ reads

$$r^2 R'' + 2r R' - \ell(\ell + 1)R = 0,$$

a Cauchy-Euler equation. The ansatz $R(r) = r^\alpha$ yields two independent solutions with $\alpha = \ell, -\ell - 1$. For the interior Dirichlet problem, we disregard the singular solution $r^{-\ell-1}$ and take $R_\ell(r) = r^\ell$.

All *azimuthally symmetric* separable solutions of interest for (66), then, are

$$u_\ell(r, \theta) = r^\ell P_\ell(\cos(\theta))$$

Superposing these solutions is sufficient for the common case of azimuthal symmetry $g(\theta, \varphi) \rightarrow g(\theta)$,

$$u(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos(\theta)), \quad (68)$$

provided that we can match any such boundary function $g(\theta)$:

$$g(\theta) = u(1, \theta, \varphi) \stackrel{?}{=} \sum_{\ell=0}^{\infty} A_\ell P_\ell(\cos(\theta))$$

As the eigenfunctions of a SL BVP, the polynomials $\{P_\ell(x)\}_{\ell=0}^{\infty}$ admit an orthogonality relation,

$$\int_{-1}^1 P_\ell(x) P_k(x) dx = \delta_{k\ell} \int_{-1}^1 (P_\ell(x))^2 dx = \frac{2}{2\ell + 1} \delta_{k\ell}.$$

Though the SL BVP is irregular, it is once again the case that for any “nice” $f(x)$ on $-1 \leq x \leq 1$,

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x) \quad \Longleftrightarrow \quad A_\ell = \frac{2\ell + 1}{2} \int_{-1}^1 f(x) P_\ell(x) dx.$$

This is the Fourier-Legendre series of $f(x)$. We may straightforwardly transform to θ :

$$g(\theta) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(\cos(\theta)) \quad \Longleftrightarrow \quad A_\ell = \frac{2\ell + 1}{2} \int_0^\pi g(\theta) P_\ell(\cos(\theta)) \sin(\theta) d\theta \quad (69)$$

With this choice of coefficients, (68) indeed solves (66) under azimuthal symmetry.

Example: Solve the BVP (66) with $g(\theta, \varphi) = g(\theta) = 1 - \cos(2\theta)$. Here, $u(r, \theta, \varphi)$ may represent the steady-state temperature of a solid ball with the surface held at this temperature distribution.

We can always resort to computing the integrals (69), but here we can deduce A_ℓ by writing

$$\begin{aligned} g(\theta) &= 1 - \cos(2\theta) = 1 - (2\cos^2(\theta) - 1) = (2 - 2x^2)|_{x=\cos(\theta)} \\ &= \left(\frac{4}{3}P_0(x) - \frac{4}{3}P_2(x) \right) |_{x=\cos(\theta)} = \frac{4}{3}P_0(\cos(\theta)) - \frac{4}{3}P_2(\cos(\theta)), \end{aligned}$$

so the only nonzero A_ℓ values are $A_0 = -A_2 = 4/3$, and the solution is

$$u(r, \theta, \varphi) = \frac{4}{3} - \frac{4}{3}r^2 P_2(\cos(\theta)) = \frac{4}{3} - \frac{2}{3}r^2(3\cos^2(\theta) - 1)$$

Visualize this solution via its level sets (e.g. surfaces of constant temperature) in Desmos [here](#). □

Day 24: Laplace's Equation in 3D: Spherical harmonics (None)

How can we expand arbitrary functions on the sphere?

Having completed the azimuthally symmetric case, we return to the general Legendre equation,

$$(1 - x^2)y'' - 2xy' + \left(\ell(\ell + 1) - \frac{m^2}{1 - x^2}\right)y = 0, \quad |y(\pm 1)| < \infty.$$

Once again, nontrivial BVP solutions exist only for $k \rightarrow \ell(\ell + 1)$ (as written), and further $\ell \geq |m|$.

These eigenfunctions are the associated Legendre polynomials³⁹ $P_{\ell m}(x)$ (of degree ℓ and order m),

$$\begin{aligned} P_{\ell m}(x) &:= (1 - x^2)^{m/2} \frac{d^m}{dx^m} [P_{\ell}(x)] \\ &= \frac{(1 - x^2)^{m/2}}{2^{\ell} \cdot \ell!} \frac{d^{\ell+m}}{dx^{\ell+m}} [(x^2 - 1)^{\ell}], \quad |m| \leq \ell = 0, 1, 2, \dots \end{aligned}$$

Our polar solutions are therefore $\Theta_{\ell m}(\theta) = P_{\ell m}(\cos(\theta))$; a number of these are listed below⁴⁰.

		m			
$P_{\ell m}(\cos(\theta))$		0	1	2	3
ℓ	0	1	0	0	0
	1	$\cos(\theta)$	$\sin(\theta)$	0	0
	2	$\frac{1}{2}(3\cos^2(\theta) - 1)$	$3\sin(\theta)\cos(\theta)$	$3\sin^2(\theta)$	0
	3	$\frac{1}{2}(5\cos^3(\theta) - 3\cos(\theta))$	$\frac{3}{2}(5\cos^2(\theta) - 1)\sin(\theta)$	$15\cos(\theta)\sin^2(\theta)$	$15\sin^3(\theta)$
			$\vdots$		

As usual, the associated Legendre polynomials $\{P_{\ell m}(x)\}_{\ell=|m|}^{\infty}$ enjoy an orthogonality relation,

$$\int_{-1}^1 P_{\ell m}(x) P_{k m}(x) dx = \delta_{k\ell} \int_{-1}^1 (P_{\ell m}(x))^2 dx = \frac{2(\ell + m)!}{(2\ell + 1)(\ell - m)!} \delta_{k\ell}.$$

We have now found all of the distinct fundamental angular solutions,

$$\Theta_{\ell m}(\theta) \Phi_m(\varphi) = P_{\ell m}(\cos(\theta)) (A \cos(m\varphi) + B \sin(m\varphi)), \quad 0 \leq m \leq \ell = 0, 1, 2, \dots$$

It is customary to collate these into the (real) spherical harmonics $Y_{\ell m}(\theta, \varphi)$, allowing $-\ell \leq m \leq \ell$:

$$Y_{\ell m}(\theta, \varphi) = \begin{cases} \sqrt{\frac{2\ell + 1}{2\pi} \frac{(\ell - |m|)!}{(\ell + |m|)!}} P_{\ell|m|}(\cos(\theta)) \sin(|m|\varphi) & m < 0 \\ \sqrt{\frac{2\ell + 1}{4\pi}} P_{\ell}(\cos(\theta)) & m = 0 \\ \sqrt{\frac{2\ell + 1}{2\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_{\ell m}(\cos(\theta)) \cos(m\varphi) & m > 0 \end{cases} \quad (70)$$

³⁹Caution: $P_{\ell m}(x)$ is only actually a polynomial for m even. For m odd, it includes a factor of $\sqrt{1 - x^2}$

⁴⁰In general, $-\ell \leq m \leq \ell$ is permitted, but in fact $P_{\ell, -m}(x) \propto P_{\ell m}(x)$, so we have only listed $P_{\ell m}(x)$ for $m \geq 0$:

$$P_{\ell, -m}(x) = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P_{\ell m}(x)$$

For instance,

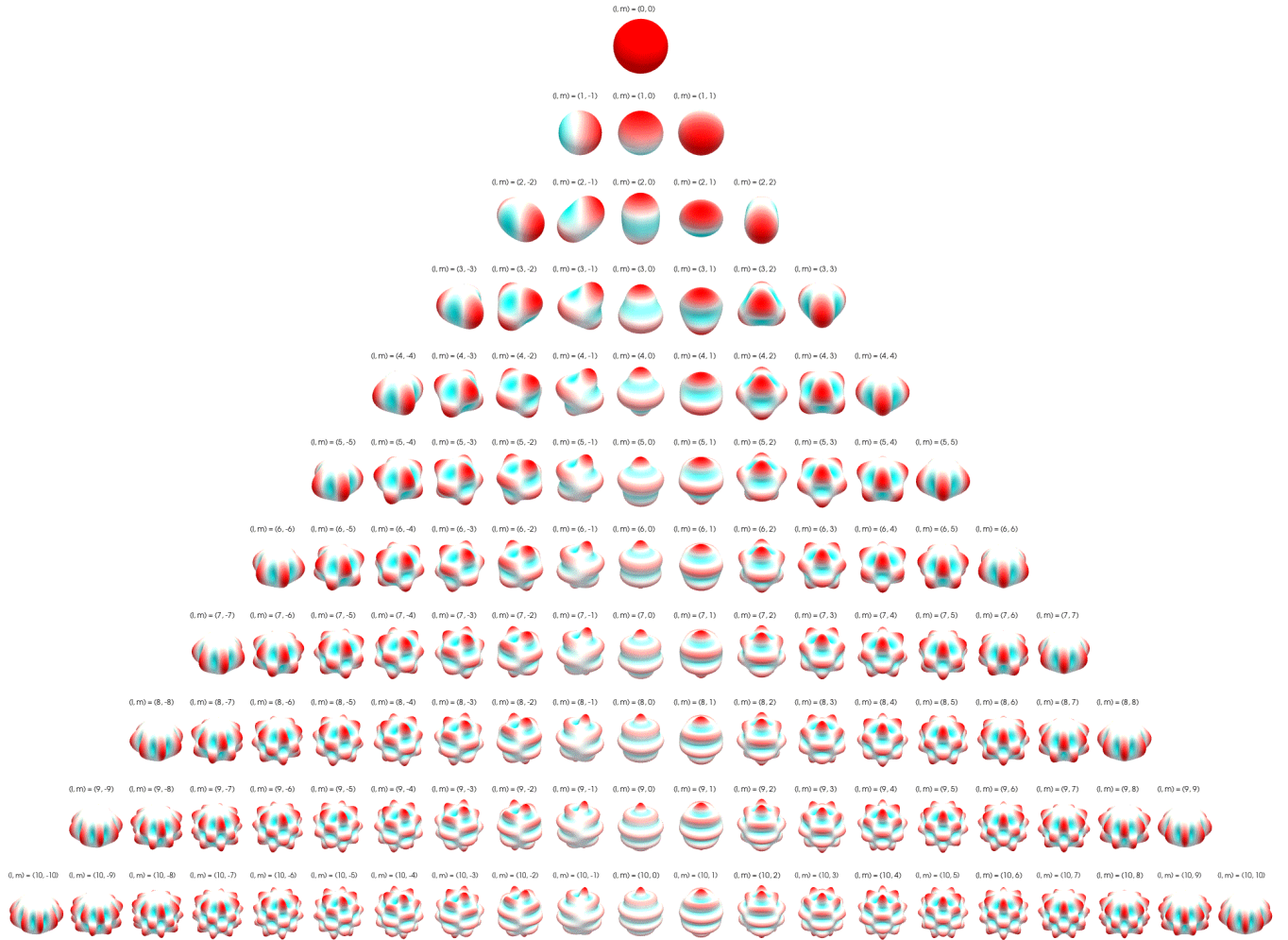
$$Y_{1,-1}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \sin(\theta) \sin(\varphi), \quad Y_{1,0}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos(\theta), \quad Y_{1,1}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \sin(\theta) \cos(\varphi).$$

These are defined so that they are orthonormal over the unit sphere:

$$\int_{S_1} Y_{\ell m}(\theta, \varphi) Y_{k n}(\theta, \varphi) dA = \int_0^{2\pi} \int_0^\pi Y_{\ell m}(\theta, \varphi) Y_{k n}(\theta, \varphi) \sin(\theta) d\theta d\varphi = \delta_{k,\ell} \delta_{n,m}$$

This makes it particularly simple to expand an angular function $g(\theta, \varphi)$ in spherical harmonics:

$$g(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta, \varphi) \quad \Longleftrightarrow \quad \boxed{A_{\ell m} = \int_{S_1} g(\theta, \varphi) Y_{\ell m}(\theta, \varphi) dA} \quad (71)$$



This image (by Wikipedia user Twistar48: see it animated [here](#)) shows the spherical harmonics $Y_{\ell m}(\theta, \varphi)$ as distortions of the sphere's radius at each (θ, φ) .

Spherical harmonics are an essential tool for parameterizing spherical functions in many fields: astronomy, cosmology, geodesy, electrostatics, quantum mechanics, chemistry, computer graphics, etc.

Finally, combining with the (interior) radial solution $R_\ell(r) = r^\ell$ yields all separable solutions,

$$u_{\ell m}(r, \theta, \varphi) = R_\ell(r)Y_{\ell m}(\theta, \varphi) = r^\ell Y_{\ell m}(\theta, \varphi),$$

and we seek the solution to (66) via superposition,

$$u(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} r^\ell Y_{\ell m}(\theta, \varphi) \quad (72)$$

Matching the BC at $r = 1$, this indeed solves (66) if $A_{\ell m}$ are chosen according to (71).

Example: Solve the BVP (66) with $g(\theta, \varphi) = 1 - \cos(2\theta) + \sin(2\theta) \sin(\varphi) - \sin^2(\theta) \cos(\theta) \cos(2\varphi)$.

We seek to express $g(\theta, \varphi)$ in terms of our angular solutions. We observe the φ dependence of each term to infer m and compare the θ dependence to the appropriate column in the table of $P_{\ell m}(\cos(\theta))$.

In the previous example, we saw that $1 - \cos(2\theta)$ gives rise to $u_1(r, \theta, \varphi) = \frac{4}{3} - \frac{2}{3}r^2(3\cos^2(\theta) - 1)$.

The term $\sin(2\theta) \sin(\varphi)$ has $m = 1$, and $\sin(2\theta) \propto P_{2,1}(\cos(\theta))$. Hence, this gives rise to $u_2(r, \theta, \varphi) = r^2 \sin(2\theta) \sin(\varphi)$.

The term $\sin^2(\theta) \cos(\theta) \cos(2\varphi)$ has $m = 2$, and $\sin^2(\theta) \cos(\theta) \propto P_{3,2}(\cos(\theta))$. Hence, this gives rise to $u_3(r, \theta, \varphi) = r^3 \sin^2(\theta) \cos(\theta) \cos(2\varphi)$. Combining,

$$u(r, \theta, \varphi) = u_1 + u_2 - u_3 = \frac{4}{3} - \frac{2}{3}r^2(3\cos^2(\theta) - 1) + r^2 \sin(2\theta) \sin(\varphi) - r^3 \sin^2(\theta) \cos(\theta) \cos(2\varphi)$$

Visualize this solution via its level sets (e.g. surfaces of constant temperature) in Desmos [here](#). □

One can solve Dirichlet problems on the exterior of a ball, or in the shell between two spheres, precisely as in 2D: include all radial solutions r^ℓ , $r^{-\ell-1}$ (shell) or just the decaying $r^{-\ell-1}$ (exterior).

Day 25: Spherical waves (None)

How do 3D waves propagate in spherical geometry?

We now consider the analogue to the circular drumhead IBVP in 3D, posed inside the unit sphere:

$$\begin{aligned} u_{tt} &= c^2 \Delta u, \\ u(1, \theta, \varphi, t) &= 0, \\ u(r, \theta, \varphi, 0) &= \phi(r, \theta, \varphi), \\ u_t(r, \theta, \varphi, 0) &= \psi(r, \theta, \varphi), \quad 0 \leq r \leq 1. \end{aligned} \tag{73}$$

Similar IBVP's describe sound or EM waves from a concentrated source, or seismic waves through the earth⁴¹. As in 2D, separating $u(r, \theta, \varphi, t) = U(r, \theta, \varphi)T(t)$ yields the Helmholtz equation,

$$\frac{T''}{c^2 T} = \frac{\Delta U}{U} = -\lambda^2 \implies \begin{aligned} \Delta U + \lambda^2 U &= 0, & U(1, \theta, \varphi) &= 0, \\ T'' + c^2 \lambda^2 T &= 0. \end{aligned}$$

As usual for waves, the T equation yields temporal oscillations,

$$T(t) = A \cos(\lambda ct) + B \sin(\lambda ct).$$

Spatially, we have an elliptic BVP. We've now seen that elliptic BVP's with zero BC often only admit $u = 0$ ⁴²: the Helmholtz eigenvalue problem seeks those special values of λ yielding otherwise.

In spherical coordinates, our elliptic BVP reads

$$\frac{1}{r^2}(r^2 U_r)_r + \frac{1}{r^2 \sin(\theta)}(\sin(\theta) U_\theta)_\theta + \frac{1}{r^2 \sin^2(\theta)} U_{\varphi\varphi} + \lambda^2 U = 0, \quad U(1, \theta, \varphi) = 0$$

We now proceed in much the same way as Day 23, separating $U(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$ to obtain

$$\begin{aligned} \frac{1}{R}(r^2 R')' + \lambda^2 r^2 + \frac{1}{\sin(\theta)Y}(\sin(\theta)Y_\theta)_\theta + \frac{1}{\sin^2(\theta)Y}Y_{\varphi\varphi} &= 0, \\ \implies \frac{1}{R}(r^2 R')' + \lambda^2 r^2 &= -\frac{1}{\sin(\theta)Y}(\sin(\theta)Y_\theta)_\theta - \frac{1}{\sin^2(\theta)Y}Y_{\varphi\varphi} = k, \end{aligned}$$

for some constant k . We now note that the angular problem is precisely that seen on Days 23-24,

$$\sin(\theta)(\sin(\theta)Y_\theta)_\theta + k \sin^2(\theta)Y + Y_{\varphi\varphi} = 0.$$

Nontrivial solutions require $k = \ell(\ell + 1)$: they are the spherical harmonics $Y_{\ell m}(\theta, \varphi)$ defined in (70).

We may now turn to a new problem, the radial equation:

$$r^2 R'' + 2r R' + (\lambda^2 r^2 - \ell(\ell + 1))R = 0, \quad R(1) = 0, \quad |R(0)| < \infty. \tag{74}$$

This is yet another irregular SL BVP. Setting $x := \lambda r$ and $R(r) =: y(x(r))$, the ODE reads

$$x^2 y'' + 2x y' + (x^2 - \ell(\ell + 1))y = 0 \tag{75}$$

⁴¹Exactly as written, this prototype problem describes, say, sound in a closed spherical chamber.

⁴²This feature is what gave us uniqueness results.

This is reminiscent of Bessel's equation. Indeed, setting $y(x) =: \frac{w(x)}{\sqrt{x}}$, so that

$$y' = \frac{w'}{\sqrt{x}} - \frac{w}{2x^{3/2}}, \quad y'' = \frac{w''}{\sqrt{x}} - \frac{w'}{x^{3/2}} + \frac{3}{4} \frac{w}{x^{5/2}},$$

this ODE (multiplied by $\sqrt{x}$) now reads

$$x^2 w'' + x w' + \left(x^2 - \left(\ell + \frac{1}{2} \right)^2 \right) w = 0.$$

This is Bessel's equation with order $\alpha = \ell + \frac{1}{2}$: w is a linear combination of $J_{\ell+1/2}, Y_{\ell+1/2}$, meaning

$$y(x) = A \frac{J_{\ell+1/2}(x)}{\sqrt{x}} + B \frac{Y_{\ell+1/2}(x)}{\sqrt{x}}$$

Due to the present context, solutions to (75) are dubbed spherical Bessel functions:

$$j_\ell(x) := \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x) \quad \text{order } \ell \text{ spherical Bessel function of the *first* kind,}$$

$$y_\ell(x) := \sqrt{\frac{\pi}{2x}} Y_{\ell+1/2}(x) \quad \text{order } \ell \text{ spherical Bessel function of the *second* kind.}$$

Note that y_ℓ diverges (and j_ℓ is finite) at $x \rightarrow 0$. Recalling the definition of $J_\alpha(x)$, consider⁴³

$$j_0(x) = \sqrt{\frac{\pi}{2x}} J_{1/2}(x) = \sqrt{\frac{\pi}{2x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \cdot \Gamma(m+3/2)} \left(\frac{x}{2} \right)^{2m+1/2} = \sqrt{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+1} \cdot m! \cdot \Gamma(m+3/2)} x^{2m}.$$

Noting that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ ⁴⁴ and $\Gamma(m+\frac{3}{2}) = (m+\frac{1}{2}) \cdot (m-\frac{1}{2}) \cdots \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(\frac{1}{2})$, the denominator is

$$\begin{aligned} 2^{2m+1} \cdot m! \cdot \Gamma\left(m+\frac{3}{2}\right) &= [2^m \cdot m!] \left[2^{m+1} \Gamma\left(m+\frac{3}{2}\right) \right] \\ &= [2m(2m-2) \cdots 4 \cdot 2] [(2m+1)(2m-1) \cdots 3 \cdot 1] \sqrt{\pi} \\ &= (2m+1)! \sqrt{\pi} \end{aligned}$$

That is, we apparently have the surprisingly simple identity

$$j_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} = \frac{\sin(x)}{x}.$$

Recall from HW8 that

$$(x^{-n} J_n(x))' = -x^{-n} J_{n+1}(x), \quad \Longleftrightarrow \quad \frac{1}{x} \frac{d}{dx} (x^{-n} J_n(x)) = -x^{-(n+1)} J_{n+1}(x)$$

One may readily check that j_ℓ inherits this identity. Iterating yields Rayleigh's formula⁴⁵:

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x} \frac{d}{dx} \right)^\ell \frac{\sin(x)}{x}$$

⁴³Recall that the Gamma function is a generalization of the factorial to non-integers, $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$.

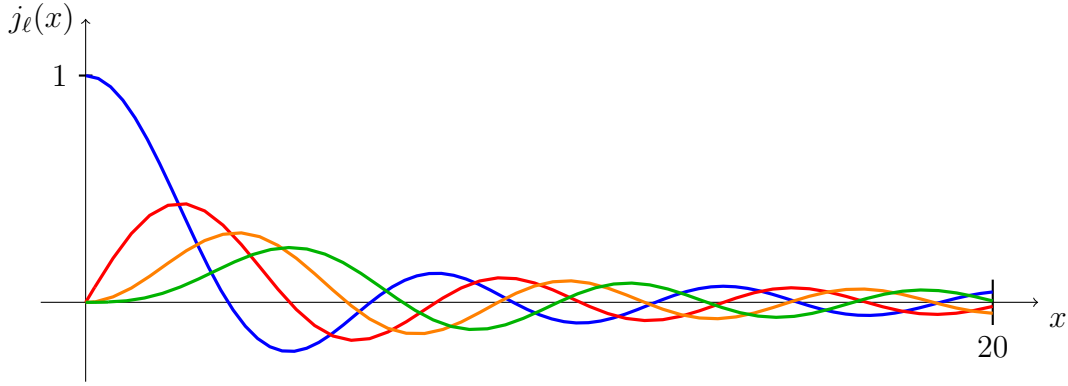
⁴⁴This is the Gaussian area $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$ in disguise.

⁴⁵While it is less useful to us, one may also find $y_\ell(x) = -(-x)^\ell \left(\frac{1}{x} \frac{d}{dx} \right)^\ell \frac{\cos(x)}{x}$

This may be used to generate any j_ℓ desired, e.g.

$$\begin{aligned} j_0(x) &= \frac{\sin(x)}{x} \\ j_1(x) &= \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \\ j_2(x) &= \left(\frac{3}{x^2} - 1\right) \frac{\sin(x)}{x} - \frac{3\cos(x)}{x^2} \\ &\vdots \end{aligned}$$

It is clear from these expressions that j_ℓ is oscillatory– the first several are plotted below.



Returning to our radial BVP (74): ODE solutions finite at $r \rightarrow 0$ are apparently $R(r) = j_\ell(\lambda r)$.

Imposing $R(1) = 0$, nontrivial solutions are $R_{\ell k}(r) = j_\ell(\lambda_{\ell k} r)$, with $\lambda_{\ell k}$ the k th positive root of j_ℓ .

These still do not admit an analytical formula in general, but they are again well-tabulated.

		ℓ				
		0	1	2	3	4
k	1	3.14	4.49	5.76	6.99	8.18
	2	6.28	7.73	9.10	10.42	11.70
	3	9.42	10.90	12.32	13.70	15.04
	4	12.57	14.07	15.51	16.92	18.30
	5	15.71	17.22	18.69	20.12	21.53
				$\vdots$		

It should come as no surprise that the functions $\{j_\ell(\lambda_{\ell k} r)\}_{k=1}^\infty$ admit an orthogonality relation,

$$\int_0^1 j_\ell(\lambda_{\ell k} r) j_\ell(\lambda_{\ell n} r) r^2 dr = \delta_{nk} \int_0^1 (j_\ell(\lambda_{\ell k} r))^2 r^2 dr = \frac{\delta_{nk}}{2} (j_{\ell+1}(\lambda_{\ell k}))^2,$$

and once again one may expand any “nice” (continuously differentiable) $f(r)$ on $0 \leq r \leq 1$,

$$f(r) = \sum_{k=1}^\infty C_k j_\ell(\lambda_{\ell k} r) \iff C_k = \frac{2}{(j_{\ell+1}(\lambda_{\ell k}))^2} \int_0^1 r^2 f(r) j_\ell(\lambda_{\ell k} r) dr.$$

Putting everything together, all spherical-separable solutions to the BVP in (73) are

$$u_{\ell m k}(r, \theta, \varphi, t) = j_\ell(\lambda_{\ell k} r) Y_{\ell m}(\theta, \varphi) [A_{\ell m k} \cos(\lambda_{\ell k} ct) + B_{\ell m k} \sin(\lambda_{\ell k} ct)],$$

These are the fundamental modes of spherical waves. A general solution may be expanded

$$u(r, \theta, \varphi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=1}^{\infty} j_{\ell}(\lambda_{\ell k} r) Y_{\ell m}(\theta, \varphi) [A_{\ell m k} \cos(\lambda_{\ell k} c t) + B_{\ell m k} \sin(\lambda_{\ell k} c t)] \quad (76)$$

One should convince themselves that this solves the IBVP (73) provided that one takes⁴⁶

$$\begin{aligned} A_{\ell m k} &= \frac{2}{(j_{\ell+1}(\lambda_{\ell k}))^2} \int_{B_1} \phi(r, \theta, \varphi) j_{\ell}(\lambda_{\ell k} r) Y_{\ell m}(\theta, \varphi) dV, \\ B_{\ell m k} &= \frac{2}{c \lambda_{\ell k} (j_{\ell+1}(\lambda_{\ell k}))^2} \int_{B_1} \psi(r, \theta, \varphi) j_{\ell}(\lambda_{\ell k} r) Y_{\ell m}(\theta, \varphi) dV \end{aligned} \quad (77)$$

Conceptually, this is the result of expanding $\phi(r, \theta, \varphi)$ (or ψ) in the spherical harmonics $Y_{\ell m}$ at each fixed r , and then expanding the resulting expansion coefficients (functions of r) in $j_{\ell}(\lambda_{\ell k} r)$.

Example: Solve the IBVP (73) with

$$\begin{aligned} \phi(r, \theta, \varphi) &= j_2(12.32r) \sin^2(\theta) \cos(2\varphi) - 0.4 j_1(4.49r) \cos(\theta) \\ \psi(r, \theta, \varphi) &= \sin(2\pi r)/r \end{aligned}$$

As usual, one could resort to the integrals (77) in general, but we can also read off the appropriate coefficients by recognizing each term as the spatial part of a separable solution:

$$\begin{aligned} j_2(12.32r) \sin^2(\theta) \cos(2\varphi) &\propto j_2(\lambda_{2,3} r) Y_{2,2}(\theta, \varphi) \\ j_1(4.49r) \cos(\theta) &\propto j_1(\lambda_{1,1} r) Y_{1,0}(\theta, \varphi) \\ \sin(2\pi r)/r &\propto j_0(\lambda_{0,2} r) Y_{0,0}(\theta, \varphi) \end{aligned}$$

Appending the appropriate temporal components for each separable solution, we have

$$u(r, \theta, \varphi, t) = j_2(12.32r) \sin^2(\theta) \cos(2\varphi) \cos(12.32ct) - 0.4 j_1(4.49r) \cos(\theta) \cos(4.49ct) + \frac{\sin(2\pi r)}{r} \frac{\sin(2\pi ct)}{2\pi c}$$

Caution: in order for this to be so straightforward, the order of j_{ℓ} must match the degree of $Y_{\ell m}$ in each term, **and** the coefficient inside j_{ℓ} (the 12.32, 4.49, and 2π above) must be a root of j_{ℓ} . $\square$

⁴⁶ B_1 denotes the unit ball in $\mathbb{R}^3$.

Day 26: Inhomogeneous Problems: Green's Functions (Lesson 36)

How can we handle sources in elliptic equations?

The prototypical inhomogeneous elliptic PDE is the Poisson equation (posed on a domain $D \subset \mathbb{R}^n$)

$$\Delta u = f,$$

for some given function f on D . Frequently, f is a *source* of some kind:

- In steady-state temperature, $-f$ is a heat source (e.g. a space heater in a room).
- In steady-state damped waves, f may be a (downward) force applied to the string/membrane.
- In electrostatics, $-f$ is the charge density sourcing the electric potential.

Indeed, a taut membrane propagating damped waves in the presence of gravity has wave equation

$$u_{tt} = c^2 \Delta u - \beta u_t - g,$$

so that the equilibrium shape of the membrane (where $u_{tt} = u_t = 0$) satisfies

$$\Delta u = \frac{g}{c^2}.$$

A powerful idea is to consider f a collection of impulses at each point in D :

$$f(\vec{x}) = \int_{\vec{y} \in D} f(\vec{y}) \delta(\vec{x} - \vec{y}) dV$$

This leads us to seek a Green's function $G(\vec{x}, \vec{x}_0)$, a response to an impulse at $\vec{x}_0 \in D$, solving

$$\Delta u = \delta(\vec{x} - \vec{x}_0).$$

By superposition: given such a G , a solution to $\Delta u = f$ will simply be

$$u(\vec{x}) = \int_{\vec{y} \in D} f(\vec{y}) G(\vec{x}, \vec{y}) dV.$$

Typically, however, we don't want just any solution: we seek the unique solution to a BVP, say

$$\Delta u = f, \quad u|_{\partial D} = g. \quad (78)$$

How can we tailor the Green's function $G(\vec{x}, \vec{x}_0)$ to particular BC's? Begin by observing

$$\int_{\partial D} \psi(\vec{\nabla} \phi) \cdot \vec{dA} = \int_D (\psi \Delta \phi + \vec{\nabla} \psi \cdot \vec{\nabla} \phi) dV$$

for any functions ϕ, ψ on D . Reversing the roles of ϕ, ψ and subtracting yields Green's second identity,

$$\int_D (\psi \Delta \phi - \phi \Delta \psi) dV = \int_{\partial D} \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) dA$$

Now let us take $\psi(\vec{y}) = u(\vec{y})$ solving (78) and $\phi(\vec{y}) = G(\vec{y}, \vec{x}_0)$ (with $\vec{x}_0$ fixed). The above reads

$$\begin{aligned} \int_{\vec{y} \in D} (u(\vec{y})\delta(\vec{y} - \vec{x}_0) - G(\vec{y}, \vec{x}_0)f(\vec{y}))dV &= \int_{\partial D} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dA \\ \implies u(\vec{x}_0) &= \int_{\vec{y} \in D} G(\vec{y}, \vec{x}_0)f(\vec{y})dV + \int_{\partial D} \left(g \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dA \end{aligned}$$

This is a consistency condition for the solution to (78) with any Green's function. If we choose the Green's function $G(\vec{x}, \vec{x}_0)$ to itself have zero Dirichlet boundary conditions, i.e. to solve the BVP

$$\Delta G = \delta(\vec{x} - \vec{x}_0), \quad G|_{\partial D} = 0, \quad (79)$$

then this consistency condition reduces⁴⁷ to an explicit formula for the u solving (78):

$$\boxed{u(\vec{x}) = \int_{\vec{y} \in D} f(\vec{y})G(\vec{x}, \vec{y})dV + \int_{\vec{y} \in \partial D} g(\vec{y})\frac{\partial G}{\partial n}(\vec{y}, \vec{x})dA} \quad (80)$$

That is: if one can somehow solve (79) for G , then one can solve *any* inhomogeneous problem (78).

The Green's function G of interest is a property of the homogeneous PDE and the geometry of D .

Even for the Laplacian, constructing G in a given geometry is rather nontrivial.

Perhaps the simplest geometry is the 2D limiting case $D \rightarrow \mathbb{R}^2$. In this limit, boundary effects are negligible, so we might seek an *isotropic* impulse response, depending only on $\|\vec{x} - \vec{x}_0\|$.

Away from $\vec{x}_0$, $G(\vec{x}, \vec{x}_0)$ should satisfy $\Delta G = 0$. From Day 22, the only isotropic solutions are

$$G(\vec{x}, \vec{x}_0) = A + B \ln(\|\vec{x} - \vec{x}_0\|)$$

What constraint is imposed by the requirement $\Delta G = \delta(\vec{x} - \vec{x}_0)$? At the very least,

$$\begin{aligned} 1 &= \int_{D_1(\vec{x}_0)} \delta(\vec{x} - \vec{x}_0)dA = \int_{D_1(\vec{x}_0)} \Delta G(\vec{x}, \vec{x}_0)dA \\ &= \int_{S_1(\vec{x}_0)} \hat{\mathbf{n}} \cdot \vec{\nabla} G(\vec{x}, \vec{x}_0)ds = \int_{S_1(\vec{x}_0)} \frac{B}{\|\vec{x} - \vec{x}_0\|} ds = 2\pi B, \end{aligned}$$

or $B = \frac{1}{2\pi}$. Being arbitrary, we may set $A = 0$, so that the full-space Green's function in 2D is

$$G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} \ln(\|\vec{x} - \vec{x}_0\|) \quad (81)$$

By (80), a solution to the Poisson equation on $\mathbb{R}^2$ is

$$u(\vec{x}) = \frac{1}{2\pi} \int_{\vec{y} \in \mathbb{R}^2} f(\vec{y}) \ln(\|\vec{x} - \vec{y}\|)dA$$

Generally, it is much more challenging to find G on a domain other than a full $\mathbb{R}^n$.

⁴⁷We've used the nontrivial fact that $G(\vec{x}, \vec{y})$ is symmetric, so $G(\vec{x}, \vec{y}) = G(\vec{y}, \vec{x})$, when G is defined by (79).

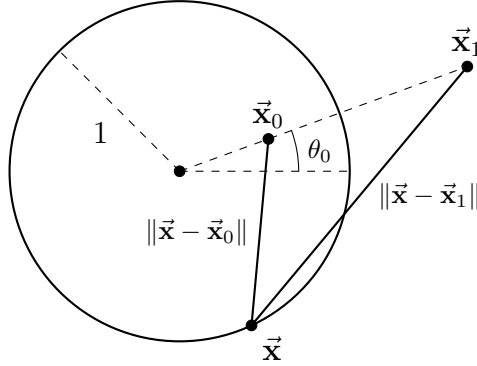
Under sufficient symmetry: the method of images combines the full-space impulse response (81) for the given $\vec{x}_0 \in D$ with the responses for a collection of other points $\vec{x}_i \notin D$ to match the BC.

Consider $D = D_1$, the unit disk in $\mathbb{R}^2$. Given $\vec{x}_0 \in D_1$, we seek an $\vec{x}_1 \notin D_1$ with the property that

$$\tilde{G}(\vec{x}, \vec{x}_0) := \frac{1}{2\pi} \ln(\|\vec{x} - \vec{x}_0\|) - \frac{1}{2\pi} \ln(\|\vec{x} - \vec{x}_1\|) = \frac{1}{2\pi} \ln\left(\frac{\|\vec{x} - \vec{x}_0\|}{\|\vec{x} - \vec{x}_1\|}\right)$$

satisfies that $\tilde{G}(\vec{x}, \vec{x}_0)$ is constant for $\vec{x} \in S_1$ (the boundary unit sphere).

That is, we require the ratio $\frac{\|\vec{x} - \vec{x}_0\|}{\|\vec{x} - \vec{x}_1\|}$ to be independent of $\vec{x} \in S_1$.



It is not obvious that such an $\vec{x}_1$ exists, but if it does, it must share $\vec{x}_0$'s polar coordinate θ_0 .

Moreover, comparing when $\vec{x}$ sits at angles θ_0 and $\theta_0 + \pi$, the radial coordinates r_0, r_1 must satisfy

$$\frac{1 - r_0}{r_1 - 1} = \frac{1 + r_0}{1 + r_1} \implies r_1 = \frac{1}{r_0}$$

We have therefore found: if a working $\vec{x}_1$ exists, it must be $\vec{x}_1 = \frac{1}{r_0^2} \vec{x}_0$. This identification gives

$$\tilde{G}(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} \ln\left(\frac{\|\vec{x} - \vec{x}_0\|}{\|\vec{x} - \frac{1}{r_0^2} \vec{x}_0\|}\right) = \frac{1}{4\pi} \ln\left(\frac{r^2 - 2rr_0 \cos(\theta - \theta_0) + r_0^2}{r^2 - 2\frac{r}{r_0} \cos(\theta - \theta_0) + \frac{1}{r_0^2}}\right),$$

with (r, θ) the polar coordinates of $\vec{x}$. For $r = 1$, this indeed reduces to a constant, $\ln(r_0^2)/4\pi$.

Subtracting this off, we've found the Green's function satisfying $G(\vec{x}, \vec{x}_0) = 0$ for $\vec{x} \in S_1$:

$$G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} \ln\left(\frac{\|\vec{x} - \vec{x}_0\|}{\|r_0 \vec{x} - \frac{1}{r_0} \vec{x}_0\|}\right) = \frac{1}{4\pi} \ln\left(\frac{r^2 - 2rr_0 \cos(\theta - \theta_0) + r_0^2}{r_0^2 r^2 - 2rr_0 \cos(\theta - \theta_0) + 1}\right) \quad (\text{see [Desmos](#)})$$

Note this is manifestly symmetric under $(r, \theta) \leftrightarrow (r_0, \theta_0)$. To fully implement (80), we must compute

$$\frac{\partial G}{\partial r}(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} \left[\frac{r - r_0 \cos(\theta - \theta_0)}{r^2 - 2rr_0 \cos(\theta - \theta_0) + r_0^2} - \frac{r_0(rr_0 - \cos(\theta - \theta_0))}{r_0^2 r^2 - 2rr_0 \cos(\theta - \theta_0) + 1} \right]$$

This is quite the mess, but it simplifies quite dramatically for $r = 1$, which is what is needed:

$$\left. \frac{\partial G}{\partial r} \right|_{r=1} = \frac{1}{2\pi} \frac{1 - r_0^2}{1 - 2r_0 \cos(\theta - \theta_0) + r_0^2}$$

Writing $\vec{y} \rightarrow \vec{x}' = (r', \theta')$ in (80), the solution to (78) for $D = D_1 \subset \mathbb{R}^2$ and BC $u(1, \theta) = g(\theta)$ is

$$u(r, \theta) = \frac{1}{4\pi} \int_{\vec{x}' \in D_1} f(r', \theta') \ln \left(\frac{r^2 - 2rr' \cos(\theta - \theta') + r'^2}{r'^2 r^2 - 2rr' \cos(\theta - \theta') + 1} \right) dA + \frac{1}{2\pi} \int_0^{2\pi} g(\theta') \frac{1 - r^2}{1 - 2r \cos(\theta' - \theta) + r^2} d\theta'$$

The first of these integrals matches the PDE inhomogeneity f , while the latter matches the BC g .

Indeed, one might recognize the latter integral as the ($R = 1$) Poisson integral formula from Day 22.

Example: One can see the output of the above formula (and compare to the homogeneous solution) for $g(\theta) = \cos(3\theta)$ and $f(r, \theta) = A$ (constant) in [this](#) Desmos demo. Physically, one should have in mind a membrane under gravity (mentioned above).

□