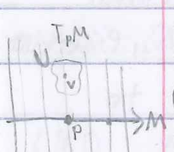
10/22

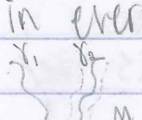
Day 3: boundary constructions, overview, ~~singularity theorems~~

We'd like to address the first question left open by our definition of a singular spacetime — where are the singularities? The GR community has made several attempts at answering this question over the past half-century by constructing boundaries associated to a spacetime (M, g).

The first of these constructions, the geodesic boundary (or g-boundary), was proposed by Geroch in 1968, motivated by the observation that if $(M, g) \subsetneq (\tilde{M}, \tilde{g})$ (same dimension) then two maximally extended (in M) geodesics $\gamma_i : (a_i, b_i) \to M$, $i = 1, 2$, limit to the same point $p \in \tilde{M} \setminus M$ as $s \to b_i$, then every nbhd $U \subset \tilde{M}$ of p has the property that each γ_i eventually enters and remains in $U \cap M$. We'd like to identify that γ_1, γ_2 "approach the same point" without reference to the extension $\tilde{M}$ (indeed we are most interested in the case that no extension exists).

Recall that to each $(p, v) \in TM$ $(:= \bigsqcup_{p \in M} \{p\} \times T_pM)$ we may associate the geodesic through p with tangent v at p. A nbhd $V \subset TM$ of (p, v) then gives rise to a <u>thickening</u> of the geodesic associated to (p, v). We might say the incomplete geodesics γ_1, γ_2 "approach the same point" if γ_1 enters and remains in every thickening of γ_2 (and vice-versa). Geroch's g-boundary is essentially the set of incomplete geodesics modulo this equivalence relation.





This construction was the first attempt at finding a set of points thought of as the "ends" of incomplete curves. While natural, it suffers the issues of nonuniqueness in its construction — there are a range of possible equivalence relations one might use that capture the essential spirit of the above — as well as the topological issue that when one appends this boundary with the natural choice of topology to the spacetime M, one finds that boundary points may not be topologically separated from interior points of M. Perhaps the largest shortcoming, however, is its inability to associate boundary points to b-incomplete non-geodesic curves. Also in 1968, Geroch found an example of a spacetime that was b incomplete but geodesically complete, so this is a necessary distinction.

In 1971, the next of the major boundary constructions, the bundle boundary (or b-boundary) was proposed by Schmidt. The motivation of this approach is that when a connected manifold has a Riemannian metric (i.e., all directions are spatial), minimal path length gives a consistent notion of distance, giving M the topological structure of a metric space, and there is a unique way to extend such a space to a larger one, called its Cauchy completion, that naturally contains the ends of all incomplete geodesics. In our Lorentzian case, we'd like to construct a related Riemannian manifold for which the Cauchy completion can yield a boundary for M.

This is precisely what Schmidt does — given the Lorentzian manifold (M, g), one can consider its frame bundle $L(M)$, essentially the set of ordered bases of all of the tangent spaces to M. This object has the structure of a smooth manifold, and one can build a Riemannian metric $\tilde{g}$ on $L(M)$ out of g having the property that if $\gamma: I \to M$ is a curve and $\{V_1, \dots, V_n\}$ is a basis of $T_{\gamma(0)}M$, then the curve $\tilde{\gamma}: I \to L(M)$ given by $\tilde{\gamma}(t) = (\gamma(t), V_1(t), \dots, V_n(t))$ (where $V_i(t)$ is the parallel propagation of V_i) has $\tilde{g}$-length

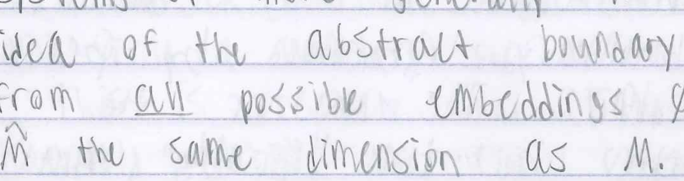
$\ell(\tilde{\gamma}) = \int_I \sum_{i=1}^{n} \alpha^i(t)^2 \, dt$, where the α^i are the expansion coefficients of $\gamma'(t)$ in the $\{V^i\}$ basis of $T_{\gamma(t)}M$, i.e. $\gamma'(t) = \sum_{i=1}^{n} \alpha^i(t) V_i(t)$. That is, $\ell(\tilde{\gamma})$ is the "Euclidean length" of γ in the $\{V_i(t)\}$ basis. The Cauchy completion of $L(M)$, then, adds the endpoints of those inextendible curves $\tilde{\gamma}$ with finite length. Translating this into the corresponding extension of M (not detailed here), this construction then adds endpoints to all of the inextendible timelike curves in M of finite proper time and bounded acceleration.

Clearly, this construction has the advantage over the g-boundary that it deals nicely with b-incompleteness rather than just geodesic incompleteness, but it is not without issues as a candidate for keeping track of singularities; in particular, it suffers some similar topological pathologies to the g-boundary's, and it does not distinguish between timelike and spacelike b-incompleteness. Further, computation requires working in the $n(n+1)$-dimensional frame bundle $L(M)$ (or at least the $\frac{n(n+1)}{2}$-dimensional orthonormal frame bundle). The b-boundary of the closed FLRW model, for example, identifies the "big bang" and "big crunch" singularities. Some modifications exist that address the topological issues, namely the so-called p-boundary (Dodson, 1978) and the projective limit alteration (Slupinski and Clarke, 1980).

The next boundary construction in our sequence is the causal boundary, or c-boundary, presented by Geroch, Kronheimer, and Penrose in 1972. The idea of the causal boundary construction is that "missing" points of the spacetime seem to give rise to past / future sets not expressible as the past / future of some $p \in M$.

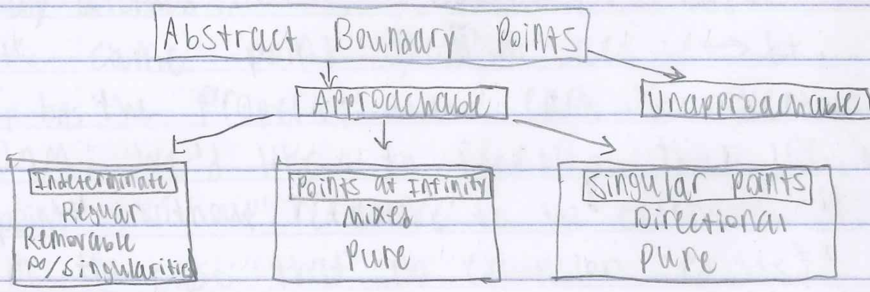
An open set $P \subseteq M$ is a past set if $P = I^-(S)$ for some subset $S \subseteq M$; similarly for future set. A past (future) set is indecomposible if it cannot be represented as the union of two distinct, proper past (future) subsets — referred to as IP's (IF's). Finally, an indecomposible past (future) set is called terminal if it is not the past (future) of any $p \in M$ — referred to as TIP's (TIF's). While it's not obvious from the definition, one can show that, when M is strongly causal, a subset $S \subseteq M$ is a TIP (TIF) iff S is the past (future) of a future- (past-) inextendible timelike curve. If we consider the set $\tilde{M}$ of IP's and IF's in M and identify $I^+(p), I^-(p) \in \tilde{M}$ with $p \in M$, $\tilde{M}$ then gives a natural extension of M wherein the new points, the TIP's and TIF's, can be thought of as endpoints of inextendible timelike curves. Again, there is an issue of non-uniqueness of the construction of a topology on this set, and the standard approaches have separation issues that are sensitive to the causal rigidity of (M, g). At face value, the c-boundary also includes points that are "at infinity", and it does not provide a natural way of distinguishing these from singular points.

Finally, we come to the most recent boundary construction, that of the abstract boundary (or a-boundary), developed by Scott and Szekeres in 1994. This construction is less explicit than those discussed so far, but it is quite versatile. While we have stressed in the past that one must be careful when trying to infer features of M from the nature of its topological boundary in some coordinate system, or more generally in an embedding $M \hookrightarrow \hat{M}$, the idea of the abstract boundary is to construct a boundary set from <u>all</u> possible embeddings $\phi : M \hookrightarrow \hat{M}$ (as smooth manifolds) with $\hat{M}$ the same dimension as M. Such triples $(M, \hat{M}, \phi)$ are called envelopments (notice no structure of a metric is involved at this stage). Under any envelopment, M has a topological boundary $\partial_\phi M \subset \hat{M}$, and we may generally consider boundary sets $B \subseteq \partial_\phi M$ in this envelopment. The abstract boundary is constructed out of all boundary subsets of all envelopments by quotienting this collection by the equivalence relation of mutual covering, where if $B \subset \partial_\phi M$, $B' \subseteq \partial_\psi M$ are boundary sets in envelopments $(M, \hat{M}, \phi)$, $(M, \hat{M}', \psi)$, we say B <u>covers</u> B' if for every nbhd $U \subseteq \hat{M}$ of B $\exists$ a nbhd $V \subseteq \hat{M}'$ of B' s.t. $\phi \circ \psi^{-1}(V \cap M) \subset U$. Written as $B \triangleright B'$.



Mutual covering, i.e. $B \triangleright B'$ and $B' \triangleright B$, gives a natural way of identifying B and B' as the "same" boundary set. An <u>abstract boundary</u> point is defined as an equivalence class of boundary sets which has a representation as a single point in some envelopment, and the <u>abstract boundary</u> is defined as the set of abstract boundary points.

So far, the abstract boundary is a property of M only as a smooth manifold. Considering the metric structure on M and endowing it with a preferred family of curves (satisfying some mild conditions — examples include the family of geodesics, C^1 curves parameterized by b-length, the timelike or causal subsets of each of these families, and even the future/past-directed timelike curve subsets of these families), one can classify the abstract boundary points according to the following scheme



Where "regular" points are those for which there exists an envelopment that extends the metric containing (a representative of) the point. If we restrict to a maximal spacetime, one for which the metric cannot be extended, the indeterminate category is no longer present, and the classification scheme reduces to identifying all approachable boundary points as singular or at infinity!

The abstract boundary approach's flexibility with respect to the choice of curve family, applicability at all levels of structure, and ability to distinguish naturally singular points from points at infinity make it an attractive construction. What's more, the topological relationship between abstract boundary points and points in M is naturally recorded in envelopments, which are necessarily Hausdorff. The biggest difficulty of this approach is, of course, practical computability, as one is faced with the task of

Knowing properties of all possible envelopments of M.